The two-point correlation function of three-dimensional O(N) models: critical limit and anisotropy.

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Abstract

In three-dimensional O(N) models, we investigate the low-momentum behavior of the two-point Green's function $G(x)$ in the critical region of the symmetric phase. We consider physical systems whose criticality is characterized by a rotational-invariant fixed point. Several approaches are exploited, such as strong-coupling expansion of lattice non-linear O(N) σ models, $1/N$-expansion, field-theoretical methods within the ϕ^4 continuum formulation.

Non-Gaussian corrections to the universal low-momentum behavior of $G(x)$ are evaluated, and found to be very small.

In non-rotational invariant physical systems with O(N)-invariant interactions, the vanishing of space-anisotropy approaching the rotational-invariant fixed point is described by a critical exponent ρ, which is universal and is related to the leading irrelevant operator breaking rotational invariance. At $N = \infty$ one finds $\rho = 2$. We show that, for all values of $N \geq 0$, $\rho \simeq 2$.

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I. INTRODUCTION

Three-dimensional $O(N)$ models describe many important critical phenomena in nature. The statistical properties of ferromagnetic materials are described by the case $N = 3$, where the lagrangian field represents the magnetization. The helium superfluid transition, whose order parameter is the complex quantum amplitude of helium atoms, corresponds to $N = 2$. The case $N = 1$ (i.e. Ising-like systems) describes liquid-vapour transitions in classical fluids or critical binary fluids, where the order parameter is the density. $O(N)$ models in the limit $N \to 0$ describe the statistical properties of long polymers.

The critical behavior of the two-point correlation function $G(x)$ of the order parameter is relevant in the description of critical scattering observed in many esperiments, typically neutron scattering in ferromagnetic materials, light and X rays in liquid-gas systems, ... In Born's approximation the cross section Γ_{fi} for incoming particles (i.e. neutrons or photons) of momentum p_i and final outgoing momentum p_f is proportional to the component $k = p_f - p_i$ of the Fourier transform of $G(x)$

$$\Gamma_{fi} \propto \widetilde{G}(k = p_f - p_i). \tag{1}$$

As a consequence of the critical behavior of the two-point function $G(x)$ at T_c,

$$\widetilde{G}(k) \sim \frac{1}{k^{2-\eta}}, \tag{2}$$

the cross section for $k \to 0$ (forward scattering) diverges as $T \to T_c$. When strictly at criticality the relation (2) holds at all momentum scales. In the vicinity of the critical point where the relevant correlation length ξ is large but finite, the behavior (2) occurs for $\Lambda \gg k \gg 1/\xi$, where Λ is a generic cut-off related to the microscopic structure of the statistical system, for example the inverse lattice spacing in the case of lattice models. At low momentum, $k \ll 1/\xi$, experiments show that $G(x)$ is well approximated by a Gaussian behavior,

$$\frac{\widetilde{G}(0)}{\widetilde{G}(k)} \simeq 1 + \frac{k^2}{M_G^2}, \tag{3}$$

where $M_G \sim 1/\xi$ is a mass scale defined at zero momentum (for a general discussion see e.g. Ref. [1]).

We will specifically consider systems with an $O(N)$-invariant Hamiltonian in the symmetric phase, where the $O(N)$ symmetry is unbroken. Furthermore, we will only consider systems with a rotationally-symmetric fixed point. Interesting members of this class are systems defined on highly symmetric lattices, i.e. Bravais or two-point basis lattices with a tetrahedral or larger discrete rotational symmetry.

In this paper we focus on the low-momentum behavior of the two-point correlation function of the order parameter, which coincides with the lagrangian field, in three-dimensional $O(N)$ models. We want to estimate the deviations from Eq. (3) in the critical region of the symmetric phase, i.e. for $0 < T/T_c - 1 \ll 1$, and in the low-momentum regime, i.e. $k^2 \lesssim M_G^2$. We focus on two quite different sources of deviations:

(i) Scaling corrections to Eq. (3), depending on the ratio k^2/M_G^2, and reflecting the non-Gaussian nature of the fixed point.

(ii) Non-rotationally invariant scaling violations, reflecting a microscopic anisotropy in the space distribution of the spins. This phenomenon may be relevant, for example, in the study of ferromagnetic materials, where the atoms lie on the sites of a lattice giving rise to a space anisotropy which may be observed in neutron-scattering experiments. In these systems the anisotropy vanishes in the critical limit, and $G(x)$ approaches a rotationally invariant form. It should be noticed that this phenomenon is different from the breakdown of the O(N) symmetry in the interaction, which has been widely considered in the literature [2].

In our study of the critical behavior of the two-point function of the order parameter $G(x)$ we will consider several approaches. We analyze the strong-coupling expansion of $G(x)$,

$$G(x) \equiv \langle \vec{s}(x) \cdot \vec{s}(0) \rangle, \tag{4}$$

for the lattice O(N) non-linear σ model with nearest-neighbor interaction

$$S_L = -N\beta \sum_{\text{links } \langle xy \rangle} \vec{s}_x \cdot \vec{s}_y, \tag{5}$$

which we have calculated up to 15th order on the simple cubic lattice and 21st order on the diamond lattice. We also perform a detailed study using the $1/N$-expansion, whose results, beside clarifying physical mechanisms, are also useful as benchmarks for the strong-coupling analysis. Moreover we compute the first few non-trivial terms of the ϵ-expansion and of the g-expansion (i.e. expansion in the four-point renormalized coupling at fixed dimension $d = 3$) of the two-point function for the corresponding ϕ^4 continuum formulation of O(N) models:

$$S_{\phi^4} = \int d^d x \ \frac{1}{2}\partial_\mu \phi(x)\partial_\mu \phi(x) + \frac{1}{2}\mu_0^2 \phi^2 + \frac{1}{4!}g_0(\phi^2)^2. \tag{6}$$

We recall that non-linear σ models and ϕ^4 models possessing the same internal symmetry O(N) describe the same critical behavior. By universality our study provides information on the behavior of the physical systems mentioned above in the critical region of the high-temperature phase. A short report of our study can be found in Ref. [3].

The first systematic study of the critical behavior of $G(x)$ is due to Fisher and Aharony [1,4,5]. They computed $G(x)$ in the ϵ-expansion up to terms $O(\epsilon^2)$ [4] and in the large-N expansion to order $1/N$ [5]; moreover some estimates of the non-Gaussian corrections for $N = 1$ and $N = 3$ were derived from strong-coupling series for various lattices [4,6]. Their calculations confirmed experimental observations that non-Gaussian corrections are small in the low-momentum region.

In this paper we reconsider the problem of determining the non-Gaussian correction to $G(x)$ in the low-momentum regime using the different approaches we mentioned above. Our strong-coupling analysis of the N-vector model on the simple cubic and diamond lattice leads to rather accurate results which considerably improve earlier calculations. This is achieved essentially for two reasons: longer strong-coupling series are available, and, more importantly, we consider improved estimators which allow more stable extrapolations to the critical limit. Our strong-coupling analysis is integrated and supported by results obtained

from the $O(1/N)$, ϵ- and g-expansion. We compute the expansion of $G(x)$ up to four-loops, $O(g^4)$, in fixed dimension $d = 3$ and we extend the results of [4] by calculating the next three-loop term $O(\epsilon^3)$. The results of the various approaches are reasonably consistent among each other: the g-expansion and the analysis of the strong-coupling series provide in general the most precise estimates, together with the $1/N$ expansion for $N \gtrsim 16$. The ϵ-expansion is somewhat worse but still consistent, perhaps because of the limited number of terms (one term less than in the g-expansion).

We also discuss the space anisotropy in $G(x)$ induced by the lattice structure. For the class of systems we consider, $G(x)$ becomes rotationally-invariant at criticality: when $\beta \to \beta_c$, so that $M_G \to 0$, the anisotropic deviations vanish as M_G^ρ, where ρ is a universal critical exponent. From a field-theoretical point of view, space anisotropy is due to non-rotationally invariant $O(N)$-symmetric irrelevant operators in the effective Hamiltonian, whose presence depends essentially on the symmetries of the physical system or of the lattice formulation. The exponent ρ is related to the critical effective dimension of the leading irrelevant operator breaking rotational invariance. On d-dimensional lattices with cubic symmetry the leading operator has canonical dimension $d+2$. In the large-N limit, where the canonical dimensions determine the scaling properties, one finds $\rho = 2$ with very small $O(1/N)$ corrections. A strong-coupling analysis supported by a two-loop ϵ-expansion and three-loop g-expansion computation indicate that ρ remains close to its canonical value for all $N \geq 0$, with deviations of approximately 1% for small values of N. It should be noted that the exponent ρ which controls the recovery of rotational invariance is different from ω, the leading subleading exponent, since they are related to different irrelevant operators. This means — and this may be of relevance for numerical calculations — that the recovery of rotational invariance is unrelated to the disappearance of the subleading corrections controlled by ω: in practice, as $\rho \approx 2$ while $0.5 \lesssim \omega \lesssim 1$ [2,7], rotational invariance is recovered long before the scaling region.

We also investigated the recovery of rotational invariance in two-dimensional models. On the square lattice, for $N = 1$ (Ising model) and $N \geq 3$, we show that $\rho = 2$. This leads us to conjecture that $\rho = 2$ holds exactly for all two-dimensional models on the square lattice. Similarly we conjecture that $\rho = 4$ (resp. $\rho = 3$) are the exact values of the exponents for the triangular (resp. honeycomb) lattice. A Monte Carlo and exact-enumeration study [8] for $N = 0$ on the square lattice is consistent with this conjecture.

We should mention that our results on space anisotropy are also relevant in the discussion of the linear response of the system in presence of an external (anisotropic) field.

The paper is organized as follows:

In Sec. II we fix the notation and introduce a general parametrization of $G(x)$ which includes the off-critical and non-spherical dependence.

In Sec. III we analyze the critical behavior of $G(x)$ at low momentum. We present calculations based on various approaches: $1/N$-expansion (up to $O(1/N)$), g-expansion (up to $O(g^4)$), ϵ-expansion (up to $O(\epsilon^3)$), and an analysis of the strong-coupling expansion of $G(x)$ on the cubic and diamond lattice.

In Sec. IV the anisotropy of $G(x)$ is studied in the critical region. We present large-N and $O(1/N)$ calculations on various lattices, and a strong-coupling analysis of some non-spherical moments of $G(x)$ on cubic and diamond lattice. Again, the analysis of the first non-trivial terms of the g-expansion and the ϵ-expansion is presented. Anisotropy in $G(x)$

is also studied in two-dimensional O(N) models.

In App. A we collect explicit formulae for the large-N limit of $G(x)$ for the nearest-neighbor N-vector model on the cubic, f.c.c. and diamond lattices.

In Apps. B and C we present some details of our $O(1/N)$ calculations.

In App. D we present the 15th-order strong-coupling expansion of the two-point function on the cubic lattice.

In App. E we report the 21st-order strong-coupling series of the magnetic susceptibility and of the second moment of $G(x)$ on the diamond lattice for $N = 1, 2, 3$.

II. THE TWO-POINT GREEN'S FUNCTION

A. Hypercubic lattices

In this Section we discuss the general behaviour of the two-point spin-spin correlation function in lattice O(N) non-linear σ models. We consider a generic Hamiltonian defined on a hypercubic lattice

$$\mathcal{H} = -\sum_{x,y} J(x-y)\vec{s}_x \cdot \vec{s}_y \tag{7}$$

where the sum runs over all lattice sites. We will later extend our analysis to other lattices. Let us define

$$\overline{k}^2(k) = 2\left[\tilde{J}(k) - \tilde{J}(0)\right] \tag{8}$$

where $\tilde{J}(k)$ denotes the Fourier transform of $J(x)$. In spite of the notation, we are not assuming that $\overline{k}^2(k)$ is a sum of the type $\sum_\mu f(k_\mu)$. We consider models for which, by a suitable normalization of the inverse temperature β, one finds

$$\overline{k}^2(k) = k^2 + O(k^4), \tag{9}$$

so that the critical limit is rotationally invariant. Moreover we make the following assumptions:

(1) The interaction $J(x)$ is short-ranged so that $\overline{k}^2$ is continuous;

(2) The function $J(x)$ (and thus also $\overline{k}^2$) is invariant under all the symmetries of the lattice;

(3) The interaction is ferromagnetic, so that $\overline{k}^2 = 0$ only for $k = 0$ in the Brillouin zone.

Beside the leading (universal) rotationally-invariant critical behaviour, we are interested in understanding the effects of the lattice structure on the two-point function and the recovery of rotational invariance. For this reason, our analysis must take into account the irrelevant operators which break rotational invariance. It is natural to expand $\overline{k}^2(k)$ in multipoles by writing

$$\overline{k}^2(k) = \sum_{l=0}^{\infty} \sum_{p=1}^{p_l} e_{2l}^{(p)}(k^2)\, Q_{2l}^{(p)}(k). \tag{10}$$

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Here the functions $Q_{2l}^{(p)}(k)$ are multipole combinations which are invariant under the symmetries of the lattice. Their expressions can be obtained from the fully symmetric traceless tensors of rank $2l$, $T_{2l}^{\alpha_1\ldots\alpha_{2l}}(k)$ [9,10], by considering all the cubic-invariant combinations, which can be obtained by setting equal an even number of indices larger than or equal to four and then summing over them. Odd-rank terms are absent in the expansion (10) because of the parity symmetry $x \to -x$. Moreover, there is no rank-two term, i.e. $Q_2(k) = 0$, due to the discrete rotational symmetry of the lattice. The summation over p in Eq. (10) is due to the fact that, for given l, there are in general many multipole combinations which are cubic invariant. The number p_l depends on the dimensionality d of the space; more precisely it is a non-decreasing function of d. It can be computed from the following generating function (for the derivation see Appendix D)

$$\sum_{l=0}^{\infty} p_l t^l = \prod_{m=2}^{d} \frac{1}{1 - t^m}. \tag{11}$$

For notational simplicity, we will suppress the explicit dependence on p in all the following formulae, but the reader should remember that it is understood in the notation.

Let us give the explicit expressions of $Q_{2l}^{(p)}(k)$ for the first few values of l. We set $Q_0(k) \equiv 1$. For $l = 2$ there is only one invariant combination, i.e. $p_2 = 1$, which can be derived from

$$T_4^{\alpha\beta\gamma\delta}(k) = k^\alpha k^\beta k^\gamma k^\delta - \frac{k^2}{(d+4)} \mathrm{Perm}_{\alpha\beta\gamma\delta}\left(\delta^{\alpha\beta} k^\gamma k^\delta\right)$$
$$+ \frac{(k^2)^2}{(d+2)(d+4)} \mathrm{Perm}_{\alpha\beta\gamma\delta}\left(\delta^{\alpha\beta}\delta^{\gamma\delta}\right), \tag{12}$$

where $\mathrm{Perm}_{\alpha_1\ldots\alpha_n}(...)$ represents the sum of the non-trivial permutations of its arguments. One then defines

$$Q_4(k) = \sum_\mu T_4^{\mu\mu\mu\mu} = k^4 - \frac{3}{d+2}(k^2)^2, \tag{13}$$

where the notation $k^n \equiv \sum_\mu k_\mu^n$ is used. For $l = 3$, $p_3 = 1$ for all $d > 2$. From

$$T_6^{\mu\nu\alpha\beta\gamma\delta}(k) = k^\mu k^\nu k^\alpha k^\beta k^\gamma k^\delta - \frac{k^2}{(d+8)} \mathrm{Perm}_{\alpha\beta\gamma\delta\mu\nu}\left(\delta^{\mu\nu} k^\alpha k^\beta k^\gamma k^\delta\right) + \tag{14}$$

$$+ \frac{(k^2)^2}{(d+6)(d+8)} \mathrm{Perm}_{\alpha\beta\gamma\delta\mu\nu}\left(\delta^{\mu\nu}\delta^{\alpha\beta} k^\gamma k^\delta\right) - \frac{(k^2)^3}{(d+4)(d+6)(d+8)} \mathrm{Perm}_{\alpha\beta\gamma\delta\mu\nu}\left(\delta^{\mu\nu}\delta^{\alpha\beta}\delta^{\gamma\delta}\right),$$

one finds

$$Q_6(k) = \sum_\mu T_6^{\mu\mu\mu\mu\mu\mu}(k) = k^6 - \frac{15k^2k^4}{d+8} + \frac{30(k^2)^3}{(d+4)(d+8)}. \tag{15}$$

In $d = 2$ it is easy to verify that $Q_6(k) = 0$ so that $p_3 = 0$. For $l = 4$ and $d > 3$ two different $Q_8^{(p)}(k)$ can be extracted from the corresponding tensor $T_8^{\alpha_1\ldots\alpha_8}$:

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$$Q_8^{(1)}(k) = \sum_\mu T_8^{\mu\mu\mu\mu\mu\mu\mu\mu}, \tag{16}$$

$$Q_8^{(2)}(k) = \sum_{\mu\nu} T_8^{\mu\mu\mu\mu\mu\nu\nu\nu}. \tag{17}$$

When $d = 2, 3$ the two combinations are not independent. Indeed $Q_8^{(2)} = 2Q_8^{(1)}$ so that $p_4 = 1$. Higher values of l can be dealt with similarly.

In order to study the formal continuum limit of the Hamiltonian defined in (7), we expand $e_{2l}(k^2)$ in powers of k^2. We write (the sum over different multipoles with the same value of l being understood in the notation)

$$\overline{k}^2(k) = \sum_{l=0}^\infty \sum_{m=0}^\infty e_{2l,m} (k^2)^m Q_{2l}(k), \tag{18}$$

where $e_{0,0} \equiv 0$ and $e_{0,1} \equiv 1$. Inserting back in Eq. (7) one sees that Eq. (18) represents an expansion in terms of the irrelevant operators

$$O_{2l,m}(x) = \vec{s}(x) \cdot \Box^m Q_{2l}(\partial)\vec{s}(x), \tag{19}$$

where $\Box \equiv \sum_\mu \partial_\mu^2$. The leading operator that breaks rotational invariance is the four-derivative term

$$O_4(x) \equiv O_{4,0}(x) = \vec{s}(x) \cdot Q_4(\partial)\vec{s}(x), \tag{20}$$

which has canonical dimensions $d + 2$.

Let us now consider the Green's function

$$G(x; \beta) \equiv \langle \vec{s}_0 \cdot \vec{s}_x \rangle, \tag{21}$$

and its Fourier transform $\widetilde{G}(k; \beta)$. We define a zero-momentum mass-scale $M_G(\beta)$ by

$$M_G(\beta) \equiv \frac{1}{\xi_G(\beta)} \tag{22}$$

where $\xi_G(\beta)$ is the second-moment correlation length

$$\xi_G^2(\beta) = \frac{1}{2d} \frac{\sum_x |x|^2 G(x; \beta)}{\sum_x G(x; \beta)}. \tag{23}$$

Since there is a one-to-one correspondence between $M_G(\beta)$ and β, one may consider $\widetilde{G}(k; \beta)$ as a function of $M_G(\beta)$ instead of β. Indeed, for the purpose of studying the critical limit, it is natural to consider $\widetilde{G}(k; \beta)$ as a function of k and M_G. In complete analogy to our discussion of $\widetilde{J}(k)$, we analyze the behaviour of $\widetilde{G}(k, M_G)$ in terms of multipoles (again a sum over different multipole combinations with the same value of l is understood, see Eq. (10)):

$$\widetilde{G}^{-1}(k, M_G) = \sum_{l=0}^\infty g_{2l}(y, M_G)Q_{2l}(k), \tag{24}$$

where $y = k^2/M_G^2$. Notice that

$$Q_{2l}(k) = Q_{2l}(k/M_G) M_G^{2l}. \tag{25}$$

For the purpose of studying the universal properties of the critical limit of $G(x)$, in which $M_G \to 0$ keeping k/M_G fixed, it is important to understand the behavior of the functions $g_{2l}(y, M_G)$ when $M_G \to 0$. The naive limit does not exist. However, as long as the contributions to $\widetilde{G}^{-1}(k, M_G)$ are originated by the insertion of individual (irrelevant) operators without any mixing among different operators with the same symmetry properties, one can apply standard results in renormalization theory. In this case one can establish some universal properties. For a generic choice of $\widetilde{J}(k)$ this holds only for the functions $g_0(y, M_G)$ and $g_4(y, M_G)$. Indeed for higher values of l there are mixings among different operators which make the renormalization of the functions $g_{2l}(y, M_G)$ more complicated. Consider, for instance, the case $l = 3$ in the large-N limit, where the operators have canonical dimensions. In this case terms proportional to $Q_4(k)$ are depressed as M_G^2, while terms proportional to $Q_6(k)$ are depressed as M_G^4. However it is easy to see that the multipole decomposition of $Q_4(k)^2$, which is also depressed as M_G^4, contains a term of the form $k^2 Q_6(k)$. This means that there are two operators contributing to $g_6(k, M_G)$, $O_{4,0}(x)^2$ and $O_{6,0}(x)$. An analogous argument applies to higher values of l. Notice that for the particular case of $l = 3$ the mixing should disappear in the limit $y \to 0$: thus for $M_G \to 0$ $g_6(0, M_G)$ can be directly related to the renormalization properties of the operator $O_{6,0}(x)$.

For $l = 0$ and $l = 2$ standard results of renormalization theory show that, if $Z_{2l}(M_G) \equiv g_{2l}(0, M_G)$, the following limit exists:

$$\lim_{M_G \to 0} \frac{g_{2l}(y, M_G)}{Z_{2l}(M_G)} = \widehat{g}_{2l}(y), \tag{26}$$

where $\widehat{g}_{2l}(y)$ is a smooth function which is normalized so that $\widehat{g}_{2l}(0) = 1$. The function $\widehat{g}_{2l}(y)$ is universal, in the sense that it is independent of the specific Hamiltonian.

The function $\widehat{g}_4(y)$ can also be obtained by considering the linear response of the system to an external field possessing the corresponding symmetry properties. One considers the one-particle irreducible two-point function with an insertion of a $O_{2l,0}(x)$ operator at zero momentum, i.e.

$$\Gamma_{O_{2l}}(x, M_G) \equiv \int dz \, \langle O_{2l,0}(z) \, \vec{s}(0) \cdot \vec{s}(x) \rangle^{irr} \tag{27}$$

and the corresponding Fourier transform $\widetilde{\Gamma}_{O_{2l}}(k, M_G)$. Setting

$$\overline{Z}_{2l}(M_G) \equiv \lim_{k \to 0} \frac{\widetilde{\Gamma}_{O_{2l}}(k, M_G)}{Q_{2l}(k)}, \tag{28}$$

the following limit exists

$$\lim_{M_G \to 0} \frac{\widetilde{\Gamma}_{O_{2l}}(k, M_G)}{\overline{Z}_{2l}(M_G)} = \widehat{g}_{2l}(y) Q_{2l}(k). \tag{29}$$

For $l = 2$ the function defined by the previous equation coincides with that defined in (26); moreover for $M_G \to 0$, $Z_4(M_G)/\overline{Z}_4(M_G)$ is a finite (non-universal) constant, meaning

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that both quantities have the same singular behavior for $M_G \to 0$. For higher values of l, formula (29) still holds, but there is no easy relation between $\widehat{g}_{2l}(y)$ and $g_{2l}(y, M_G)$ as defined in Eq. (24), at least for generic Hamiltonians. Indeed, at least in principle, one may consider specific forms of $\widetilde{J}(k)$ enjoying the property that all contributions $g_{2n}(k, M_G)$ with $0 < n < \bar{l}$ vanish in the critical limit, for a given value of $\bar{l}$. In lattice quantum field theory this is essentially the spirit of Symanzik's improvement program [11]. In this case formula (26) is valid for $l = \bar{l}$ and the corresponding function $\widehat{g}_{2l}(y)$ coincides with that defined by Eq. (29).

The functions $\widehat{g}_{2l}(y)$ defined in Eq. (29) have a regular expansion in y around $y = 0$:

$$\widehat{g}_{2l}(y) = 1 + c_{2l,1}y + c_{2l,2}y^2 + \ldots. \tag{30}$$

$c_{0,1} = 1$ due to the definition of the second-moment correlation length.

The renormalization constant $\overline{Z}_{2l}(M_G)$ is instead non-universal. For $M_G \to 0$ it behaves as

$$\overline{Z}_{2l}(M_G) \approx \overline{z}_{2l}M_G^{-\eta_{2l}}, \tag{31}$$

where η_{2l} is a critical exponent which depends only on the spin of the representation (i.e. it does not depend on the additional index p which has always been understood in the notation, see Eq. (10)), and $\overline{z}_{2l}$ is a non-universal constant which depends on the lattice and on the Hamiltonian (and the additional index p). An analogous expression is valid for $Z_4(M_G)$ (and for $Z_{2\bar{l}}$ for the special Hamiltonians we have discussed before): for $M_G \to 0$ we have $Z_4(M_G) \approx z_4 M_G^{-\eta_4}$. For $l = 0$, as a consequence of our definitions,

$$Z_0(M_G) \sim M_G^{2-\eta}, \tag{32}$$

where η is the standard anomalous dimension of the field. More generally $\sigma_{2l} \equiv \eta - \eta_{2l}$ is the anomalous dimension of the irrelevant operator $O_{2l,0}(x)$.

In two dimensions the renormalization constants diverge only logarithmically and thus we write for $l \neq 0$

$$\overline{Z}_{2l}(M_G) \approx \overline{z}_{2l}(\ln M_G)^{\gamma_{2l}}\left[1 + O\left(\frac{1}{\ln M_G}\right)\right]. \tag{33}$$

The anomalous dimensions γ_{2l} are universal while the prefactor $\overline{z}_{2l}$ depends on the details of the interaction.

We can now discuss the critical limit of Eq. (24). Using the previous formulae we can write for $M_G \to 0$

$$\frac{\widetilde{G}^{-1}(k, M_G)}{Z_0(M_G)} \approx \widehat{g}_0(y) + \text{``rot. inv. sublead.''} + \frac{z_4}{z_0}M_G^{2+\eta-\eta_4}\widehat{g}_4(y)Q_4(k/M_G) + \ldots \tag{34}$$

where "rot. inv. sublead." indicates rotationally-invariant subleading corrections and the dots stand for terms which vanish faster as $M_G \to 0$. From Eq. (34) one immediately convinces oneself that the anisotropic effects in $G(x)$ vanish for $M_G \to 0$ as M_G^ρ where ρ is a universal critical exponent given by

$$\rho = 2 + \eta - \eta_4. \tag{35}$$

We must notice that the exponent ρ is not related to the exponent ω which characterizes the critical behaviour of the "subleading" terms which vanish as M_G^ω, as they are connected to different (rotationally-invariant) irrelevant operators. Finally notice that the leading term breaking rotational invariance is universal apart from a multiplicative constant, the factor z_4/z_0.

Let us now consider the small-momentum limit in which $y \to 0$ keeping M_G fixed. In this case one can write for $l = 0, 2$ (or in the special case we have discussed above for $l = 0, \bar{l}$)

$$g_{2l}(y, M_G) = \sum_{m=0}^{\infty} u_{2l,m}(M_G)y^m. \tag{36}$$

By comparing this expansion with Eq. (30) and using Eq. (26), one recognizes that

$$Z_{2l}(M_G) = u_{2l,0}(M_G), \tag{37}$$

and

$$c_{2l,m} = \lim_{M_G \to 0} \frac{u_{2l,m}(M_G)}{u_{2l,0}(M_G)}. \tag{38}$$

In the following sections we will use this formula to derive estimates for $c_{2l,m}$. Indeed the functions $u_{2l,m}(M_G)$ can be determined by computing dimensionless invariant ratios of moments of $G(x; \beta)$:

$$q_{2l,m}(\beta) = \sum_x (x^2)^m Q_{2l}(x) G(x; \beta). \tag{39}$$

It is interesting to notice that the expansion (34) implies some universality properties for some ratios of $q_{2l,m}$. It is easy to verify that

$$R_{4,m,n}(\beta) = \frac{q_{0,n}(\beta)q_{4,m}(\beta)}{q_{0,m}(\beta)q_{4,n}(\beta)} \tag{40}$$

is universal for $T \to T_c$; indeed the constant z_4/z_0 drops out in the ratio. Notice that this means that not only $R_{4,m,n}$ does not depend on the particular Hamiltonian, but also that it is independent of the lattice structure as long as $O_{4,0}(x)$ is the leading operator breaking rotational invariance.

B. Other regular lattices

All the considerations of the previous subsection can be extended without changes to other lattices with cubic symmetry, such as the b.c.c. and the f.c.c. lattices. For other Bravais lattices the same general formulae hold, but different multipole combinations will appear in the expansion, according to the symmetry of the lattice. In general a larger number of multipole combinations with given spin appears when considering lattices with a lower symmetry. It is important to notice that in order to have a rotationally-invariant critical

limit no multipole $Q_l(k)$ with $l = 2$ should appear in the expansion of the Hamiltonian. Thus our considerations apply only to highly symmetric lattices with a tetrahedral or larger discrete rotational group.

As an example of a non-cubic-like lattice let us consider the two-dimensional triangular lattice. It is invariant under rotation of $\pi/3$. The relevant multipoles are

$$T_{6l}(k) = (-k^2)^{3l} \cos(6l\theta) = \sum_{m=0}^{3l} \binom{6l}{2m} k_y^{2m} (ik_x)^{6l-2m}, \tag{41}$$

where we have set $k_x = |k| \cos\theta$, $k_y = |k| \sin\theta$ and we have assumed one of the generators of the lattice to be parallel to the x-axis. Thus in this case we write

$$\overline{k}^2(k) = \sum_{l=0}^{\infty} T_{6l}(k) e_{6l}(k^2), \tag{42}$$

and a similar expression for the expansion of the two-point function. For the triangular lattice the first operator which breaks rotational invariance has dimension $d + 4$. This is a consequence of the fact that the triangular lattice has a larger symmetry group with respect to the square lattice. We define moments corresponding to $T_{6l}(k)$ by

$$t_{6l,m}(\beta) = \sum_x (x^2)^m T_{6l}(x) G(x; \beta). \tag{43}$$

The arguments given in the previous subsection can be generalized to the triangular lattice in a straightforward way. One derives an expansion of the form (34) with $\rho = 4 + \eta - \eta_6$, $T_6(k/M_G)$ and $\hat{g}_6(y)$ substituting $Q_4(k/M_G)$ and $\hat{g}_4(y)$.

C. Non-Bravais lattices

Up to now we have considered regular (Bravais) lattices. However other important lattice structures are represented by lattices with basis. Particular examples are the honeycomb lattice in two dimensions and the diamond lattice in three dimensions. These lattices are generically defined by the set of points $\vec{x}$ such that

$$\vec{x} = \vec{x}' + p\vec{\eta}_p, \qquad \vec{x}' = \sum_i l_i \vec{\eta}_i, \tag{44}$$

where $\vec{\eta}_p$ is the so-called basis vector joining the two points of the basis, and $\vec{\eta}_i$ are the generators of the underlying regular lattice. Here $p = 0, 1$ and $l_i \in Z$. For the honeycomb lattice $\vec{\eta}_i$ are the generators of a triangular lattice while for the diamond lattice $\vec{\eta}_i$ are the generators of a f.c.c. lattice. Due to the breaking of translational invariance one distinguishes between correlations between points with the same value of p (i.e. points belonging to the same regular lattice) and points with different p. In general the components $G_{pp'}$ of the two-point correlation function can be written in the form

$$G_{00}(x - y) = G_{11}(x - y) = \int \frac{dk}{V_B} e^{ik(x-y)} \frac{1}{\Delta(k, M_G)} \tag{45}$$

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and

$$G_{01}(x - y) = G_{10}(y - x) = \int \frac{dk}{V_B} e^{ik(x-y)} \frac{H(k, M_G)}{\Delta(k, M_G)}, \tag{46}$$

where the integrals are performed over the Brillouin zone of the corresponding underlying regular lattice, V_B being its volume. $G_{11}(x)$ and therefore $\Delta(k, M_G)$ have the symmetries of the underlying regular lattice and thus can be expanded as in the first subsection. On the other hand, $H(k, M_G)$ does not have the symmetry of the regular lattice, but only the reduced symmetry of the full lattice. For the Gaussian model with nearest-neighbor interactions defined on the honeycomb and diamond lattices (and also on their d-dimensional generalization), it is easy to realize that, when $M_G \to 0$,

$$\Delta(k, M_G) \longrightarrow d\left[1 - |H(k, 0)|^2\right] + M_G^2, \tag{47}$$

and $\Delta(k, M_G)$ turns out to be the inverse propagator for the Gaussian theory defined on the corresponding regular lattice. In App. A 3 we present a more detailed analysis of the Gaussian theory with nearest-neighbor interactions on the diamond lattice.

Because of the reduced symmetry, additional multipoles which are not parity-invariant appear in the expansion of $H(k, M_G)$. In the case of the honeycomb lattice the symmetry of the triangular lattice is reduced to $\theta \to \theta + \frac{2\pi}{3}$. Assuming that one of the links leaving a site is parallel to the x-axis, one can write

$$H(k, M_G) = \sum_{l=0}^{\infty} T_{3l}(k) h_{3l}(y, M_G), \tag{48}$$

where we have extended the definition (41) to include odd multipoles:

$$T_{3l}(k) = (-k^2)^{3l/2} \cos(3l\theta) = \sum_{m=0}^{3l/2} \binom{3l}{2m} k_y^{2m} (ik_x)^{3l-2m}. \tag{49}$$

The factor i in this equation insures that the functions $h_{3l}(y, M_G)$ are real for all l.

For the diamond lattice one can write

$$H(k, M_G) = \sum_{l=0}^{\infty} \sum_{p=1}^{p_l} Q_l^{(p)}(k) h_l^{(p)}(y, M_G), \tag{50}$$

where $Q_l^{(p)}(k)$ are multipoles constructed from $T_l^{\alpha_1 \cdots \alpha_l}$ as in the case of the cubic lattices. The only difference is that now odd-spin operators are allowed, belonging to the class

$$Q_{2l+3}(k) = ik_1 k_2 k_3 Q_{2l}(k), \tag{51}$$

where we have assumed the natural orientation of the underlying f.c.c. lattice (see App. A 3).

For these lattices, it is not straightforward to make contact with the field-theoretical approach. The problem is writing down operators in the effective Hamiltonian that break the parity symmetry. These operators must have an odd number of derivatives, but, if they are bilinear in a real field ϕ, they give after integration only boundary terms. The solution

to this apparent puzzle comes from the fact that the effective Hamiltonian for models on lattices with basis is naturally written down in terms of two fields, defined on the two regular sublattices (for the diamond lattice see App. A 3).

As in the regular lattice case, we can associate to the breaking of the parity symmetry a universal exponent ρ_p. In principle it can be derived from the critical dimension of the lower-dimensional operator breaking this symmetry. From a practical point of view it is simpler to consider moments of $G(x)$. For the diamond lattice one defines ρ_p from the behavior, for $M_G \to 0$, of the odd moments $q_{3,m}(\beta)$, i.e.

$$\frac{q_{3,m}(\beta)}{q_{0,0}(\beta)} \sim M_G^{-3-2m+\rho_p}. \tag{52}$$

The same formula applies to the honeycomb lattice with the obvious substitutions, $q_{0,0} \to t_{0,0}$, $q_{3,m} \to t_{3,m}$.

III. CRITICAL BEHAVIOR OF $G(x)$ AT LOW MOMENTUM

A. Parametrization of the spherical limit of $G(x)$ at low momentum

According to the discussion presented in the previous section, in the critical limit multipole contributions are depressed by powers of M_G, hence for $\beta \to \beta_c$

$$\frac{\widetilde{G}(0; \beta)}{\widetilde{G}(k; \beta)} \longrightarrow \widehat{g}_0(y). \tag{53}$$

where, again, $y = k^2/M_G^2$. As already stated by Eq. (30), $\widehat{g}_0(y)$ can be expanded in powers of y around $y = 0$:

$$\widehat{g}_0(y) = 1 + y + \sum_{i=2}^{\infty} c_i y^i, \tag{54}$$

where $c_i \equiv c_{0,i}$. For generalized Gaussian theories $c_i = 0$. As discussed in Sec. II A the coefficients c_i of the low-momentum expansion of $\widehat{g}_0(y)$ can be related to the critical limit of appropriate dimensionless ratios of spherical moments

$$m_{2j} \equiv q_{0,j} = \sum_x |x|^{2j} G(x), \tag{55}$$

or of the corresponding weighted moments

$$\overline{m}_{2j} \equiv \frac{m_{2j}}{m_0}. \tag{56}$$

It is easy to compute the behavior of $\overline{m}_{2j}$ for $M_G \to 0$:

$$\begin{aligned} \overline{m}_2 &\approx 2dM_G^{-2}, \\ \overline{m}_4 &\approx 8d(d+2)(1-c_2)M_G^{-4}, \\ \overline{m}_6 &\approx 48d(d+2)(d+4)\left(1-2c_2+c_3\right)M_G^{-6}, \end{aligned} \tag{57}$$

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etc.., where d is the lattice dimension. Then, introducing the quantities

$$v_{2j} = \frac{1}{2^j j! \prod_{i=0}^{j-1}(d+2i)} \overline{m}_{2j} M_G^{2j},$$
(58)

one may compute $\hat{u}_i \equiv u_{0,i}/u_{0,0}$ from the following combinations of v_{2j}

$$\begin{aligned}
\hat{u}_2 &= 1 - v_4, \\
\hat{u}_3 &= 1 - 2v_4 + v_6,
\end{aligned}$$
(59)

etc... By definition, see Eqs. (38) and (54), in the critical limit $\hat{u}_i \to c_i$.

Another important quantity which characterizes the low-momentum behavior of $\hat{g}_0(y)$ is the critical limit of the ratio M^2/M_G^2,

$$S_M \equiv \lim_{\beta \to \beta_c} \frac{M^2}{M_G^2},$$
(60)

where M is the mass-gap of the theory, that is the mass determining the long-distance exponential behavior of $G(x)$. The value of S_M is related to the negative zero y_0 of $\hat{g}_0(y)$ which is closest to the origin by

$$y_0 = -S_M.$$
(61)

The constant S_M is one in Gaussian models (i.e., when $\hat{g}_0(y) = 1+y$), like the large-N limit of O(N) models.

Let us now consider the relation between the renormalization constant Z_G defined at zero momentum,

$$Z_G \equiv \chi M_G^2 = Z_0^{-1} M_G^2,$$
(62)

where Z_0 has been introduced in Eq. (26), and the on-shell renormalization constant Z, which is defined by

$$\tilde{G}(k) \longrightarrow \frac{Z}{M^2 + k^2}$$
(63)

when $k \to iM$. The mass gap M and the constant Z determine the large-distance behavior of $G(x)$; indeed for $|x| \to \infty$

$$G(x) \longrightarrow \frac{Z}{2M} \left(\frac{M}{2\pi|x|}\right)^{\frac{d-1}{2}} e^{-M|x|}.$$
(64)

The critical limit S_Z of the ratio Z_G/Z is a universal quantity given by

$$S_Z = \lim_{\beta \to \beta_c} \frac{Z_G}{Z} = \frac{\partial}{\partial y} \hat{g}_0(y)|_{y=y_0}.$$
(65)

In a Gaussian theory $Z_G = Z$.

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B. $1/N$-expansion

In the large-N limit the difference

$$\widehat{g}_0(y) - (1 + y) \tag{66}$$

is depressed by a factor $1/N$. It can be derived from the $1/N$ expansion of the self-energy in the continuum formulation. One finds [5,12]

$$\widehat{g}_0(y) = 1 + y + \frac{1}{N}\phi_1(y) + O\left(\frac{1}{N^2}\right). \tag{67}$$

where, for $d = 3$,

$$\phi_1(y) = \frac{2}{\pi} \int_0^\infty dz \frac{z}{\arctan\left(\frac{1}{2}\sqrt{z}\right)} \left[\frac{1}{4\sqrt{yz}} \ln\left(\frac{y + z + 2\sqrt{yz} + 1}{y + z - 2\sqrt{yz} + 1}\right) - \frac{1}{z+1} + \frac{y(3-z)}{(z+1)^3} \right]. \tag{68}$$

A general discussion of the $O(1/N)$ correction to $\widehat{g}_0(y)$ in d-dimension is presented in App. B. In particular Eq. (68) can be derived from Eqs. (B1), (B2) and (B12). The coefficients c_i of the low-momentum expansion of $\widehat{g}_0(y)$ turn out to be very small. Writing them as

$$c_i = \frac{c_i^{(1)}}{N} + O\left(\frac{1}{N^2}\right), \tag{69}$$

one obtains

$$
\begin{aligned}
c_2^{(1)} &= -0.00444860..., \\
c_3^{(1)} &= 0.000134410..., \\
c_4^{(1)} &= -0.0000065805..., \\
c_5^{(1)} &= 0.0000004003...,
\end{aligned}
\tag{70}
$$

etc.. For sufficiently large N we then expect

$$c_i \ll c_2 \ll 1 \qquad \text{for} \quad i \geq 3. \tag{71}$$

As we shall see from the analysis of the strong-coupling expansion of $G(x)$, the pattern (71) is verified also at low values of N.

The ratio $S_M \equiv M^2/M_G^2$ is obtained by evaluating the negative zero y_0 of $\widehat{g}_0(y)$ closest to the origin:

$$S_M = -y_0 = 1 + \frac{1}{N}\phi_1(-1) + O\left(\frac{1}{N^2}\right), \tag{72}$$

where

$$\phi_1(-1) = -0.00459002.... \tag{73}$$

Moreover using Eq. (65), one finds

$$S_Z = 1 + \frac{1}{N}\phi_1'(-1) + O\left(\frac{1}{N^2}\right) \tag{74}$$

where

$$\phi_1'(-1) = 0.00932894.... \tag{75}$$

As expected from the relations (71) among the coefficients c_i, a comparison with Eqs. (70) shows that the non-Gaussian corrections to S_M and S_Z are essentially determined by the term proportional to $(k^2)^2$ in $\tilde{G}^{-1}(k)$, through the approximate relations

$$S_M \simeq 1 + c_2, \tag{76}$$
$$S_Z \simeq 1 - 2c_2, \tag{77}$$

with corrections of $O(c_3)$.

C. g-expansion in three dimensions

Another approach to the study of the critical behavior in the symmetric phase of $O(N)$ models is based on the so-called g-expansion, the perturbative expansion at fixed dimension $d = 3$ for the corresponding ϕ^4 continuum formulation [13]. The perturbative series which are obtained in this way are asymptotic; nonetheless accurate results can be obtained using a Borel transformation and a conformal mapping which take into account their large-order behavior. As general references on this method see for instance Refs. [2] and [14]. This technique has led to very precise estimates of the critical exponents.

Starting from the continuum action (6), one renormalizes the theory at zero momentum using the following renormalization conditions for the irreducible two- and four-point correlation functions of the field ϕ:

$$\Gamma^{(2)}(p)_{\alpha\beta} = Z_G^{-1}\Gamma_R^{(2)}(p)\,\delta_{\alpha\beta}, \tag{78}$$
$$\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} = -Z_G^{-2}\frac{g}{3}M_G\,\delta_{\alpha\beta\gamma\delta}, \tag{79}$$

where

$$\Gamma_R^{(2)}(p) = M_G^2 + p^2 + O(p^4), \tag{80}$$

and $\delta_{\alpha\beta\gamma\delta} \equiv \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}$. When $M_G \to 0$ the renormalized coupling constant is driven toward an infrared stable zero g^* of the β-function

$$\beta(g) \equiv M_G\frac{\partial g}{\partial M_G}|_{g_0,\Lambda}\, . \tag{81}$$

The universal function $\widehat{g}_0(y)$ is related to the renormalized function

$$f(g,y) \equiv M_G^{-2}\Gamma_R^{(2)}(k) \tag{82}$$

by

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$$\widehat{g}_0(y) = \lim_{g \to g^*} f(g, y). \tag{83}$$

We computed the first three non-trivial orders of the non-Gaussian corrections to $\widehat{g}_0(y)$. A calculation up to four loops gave

$$f(g, y) = 1 + y + \bar{g}^2 Z_g^2 Z_G \frac{N+2}{(N+8)^2} \varphi_2(y) + \bar{g}^3 Z_g^3 Z_G^{3/2} \frac{N+2}{(N+8)^2} \varphi_3(y) + \tag{84}$$

$$\bar{g}^4 Z_g^4 Z_G^2 \frac{N+2}{(N+8)^2} \left[\frac{(N+2)}{(N+8)^2} \varphi_{4,1}(y) + \frac{(N^2+6N+20)}{(N+8)^2} \varphi_{4,2}(y) + \frac{(5N+22)}{(N+8)^2} \varphi_{4,3}(y) \right] + O(\bar{g}^5),$$

where $\bar{g}$ is the rescaled coupling [2]

$$\bar{g} = \frac{(N+8)}{48\pi} g, \tag{85}$$

Z_g is the renormalization constant of the coupling (defined by $g_0 = M_G g Z_g$)

$$Z_g = 1 + \bar{g} + \left[1 - \frac{2(41N+190)}{27(N+8)^2} \right] \bar{g}^2 + O(\bar{g}^3), \tag{86}$$

and Z_G is the zero-momentum renormalization of the field

$$Z_G = 1 - \frac{4(N+2)}{27(N+8)^2} \bar{g}^2 + O(\bar{g}^3). \tag{87}$$

A simple derivation of the two and three-loop functions $\varphi_2(y)$ and $\varphi_3(y)$ is presented in App. B (cfr. Eqs. (B19)). In particular using the results of Refs. [15,16] one finds

$$\varphi_2(y) = 4\ln(1 + \tfrac{1}{9}y) + 24\frac{\arctan(\sqrt{y}/3)}{\sqrt{y}} - 8 - \frac{4}{27}y. \tag{88}$$

We shall not report the expressions of the four-loop functions $\varphi_{4,j}(y)$ because they are not very illuminating.

The coefficients of the low-momentum expansion can be easily obtained from Eq. (85) by calculating the zero-momentum derivatives of the functions $\varphi_{n,j}(y)$. We write

$$c_i = \frac{N+2}{(N+8)^2} h_i^{(2)} \bar{g}^2 + \frac{N+2}{(N+8)^2} \left(2h_i^{(2)} + h_i^{(3)} \right) \bar{g}^3 + \frac{N+2}{(N+8)^2} \left\{ 2h_i^{(2)} \left[1 - \frac{8(7N+32)}{3(N+8)^2} \right] \right.$$

$$\left. + 3h_i^{(3)} + h_i^{(4,1)} \frac{(N+2)}{(N+8)^2} + h_i^{(4,2)} \frac{(N^2+6N+20)}{(N+8)^2} + h_i^{(4,3)} \frac{(5N+22)}{(N+8)^2} \right\} \bar{g}^4 + O(\bar{g}^5), \tag{89}$$

where we have introduced the coefficients

$$h_i^{(n,j)} = \frac{1}{i!} \frac{d^i}{dy^i} \varphi_{n,j}(y)|_{y=0}. \tag{90}$$

In Table I we report the numerical values of $h_i^{(k,j)}$ for $i \leq 5$.

By evaluating the zero of $\widehat{g}_0(y)$ closest to the origin, one obtains

$$S_M = 1 + \bar{g}^2 \frac{N+2}{(N+8)^2} \varphi_2(-1) + \bar{g}^3 \frac{N+2}{(N+8)^2} \left[2\varphi_2(-1) + \varphi_3(-1)\right] + O(\bar{g}^4), \tag{91}$$

which numerically leads to

$$S_M - 1 = -\bar{g}^2 \frac{N+2}{(N+8)^2} 0.00521783 \left[1 + \bar{g} \times 0.054182 + O(\bar{g}^2)\right]. \tag{92}$$

Moreover

$$S_Z = 1 + \bar{g}^2 \frac{N+2}{(N+8)^2} \varphi_2'(-1) + \bar{g}^3 \frac{N+2}{(N+8)^2} \left[2\varphi_2'(-1) + \varphi_3'(-1)\right] + O(\bar{g}^4), \tag{93}$$

which numerically leads to

$$S_Z - 1 = \bar{g}^2 \frac{N+2}{(N+8)^2} 0.0107349 \left[1 + \bar{g} \times 0.041829 + O(\bar{g}^2)\right]. \tag{94}$$

A comparison of the g-expansions of c_i, S_M and S_Z shows that the approximate relations (76) and (77) are valid for all values of N and not only for $N \to \infty$ as shown in the previous subsection.

In order to get quantitative estimates, one must perform a resummation of the series and then evaluate it at the fixed-point value of the coupling $\bar{g}^*$. Although the terms of the g-expansion we have calculated are only three for c_i and two for S_M and S_Z, we have tried to extract quantitative estimates taking into account also the following facts:

(i) The g-expansion is Borel-summable [17] (see also e.g. Refs. [2] and [14] for a discussion of this issue), and the singularity closest to the origin of the Borel transform (corresponding to the rescaled coupling $\bar{g}$) is known [18]: $b_s = -0.75189774 \times (N+8)$.

(ii) The fixed point value $\bar{g}^*$ of $\bar{g}$ has been accurately determined by analyzing a much longer expansion (to $O(g^7)$) of the corresponding β-function [19–22]. Indipendent and consistent estimates of $\bar{g}^*$ have been obtained by other approaches, such as strong-coupling expansion of lattice non-linear O(N) σ models [23,24] (for $N = 1$ see also Refs. [25–28]), and Monte Carlo lattice simulations (only data for $N = 1$ are available [29–32]).

We have followed the procedure described in Ref. [33] (see also Ref. [2]), where the perturbative expansion in powers of $\bar{g}$ is summed using a Borel transformation and a conformal mapping which takes into account its large-order behavior. We transform the series

$$R(g) = \sum_{k=0} R_k g^k \tag{95}$$

into

$$R(g) = \sum_{k=0} B_k \int_0^\infty e^{-t} u(gt)^k dt \tag{96}$$

where

$$u(x) = \frac{\sqrt{x - b_s} - \sqrt{-b_s}}{\sqrt{x - b_s} + \sqrt{-b_s}}. \tag{97}$$

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The coefficients B_k are determined by comparing the expansion in powers of g of the r.h.s. of Eq. (96) with the original expansion. Since the g-series of c_i, S_M-1 and S_Z-1, have the form $R(\bar{g}) = \bar{g}^2 \sum_{k=0} a_i \bar{g}^i$, one may apply the resummation method to either $R(\bar{g})$ or $R(\bar{g})/\bar{g}^2$. In Table V we present results for both choices. Following the suggestions of Ref. [33] we also tried more refined resummations, changing formula (96) to weaken the singularity of the Borel transform. We did not find any significant difference. In our calculations we used the estimates of $\bar{g}^*$ obtained from the analysis of the β-function by [19,20,22]. They are reported in Table II. For small values of N slightly lower values of $\bar{g}^*$ were computed in Ref. [34] taking into account the non-analiticity of the β-function at the critical point [21]. This difference is however too small to be quantitatively relevant in our calculations.

It is difficult to estimate the uncertainty of the results: the fluctuations of the results with respect to the method we used to resum the perturbative series indicate an error of $\lesssim 20\%$ on c_i and S_M for small values of N. As N increases the estimates become more stable. The final results are in good agreement with the estimates by other methods.

D. ϵ-expansion

The universal function $\hat{g}_0(y)$ can be computed perturbatively in $\epsilon = 4 - d$ using the continuum ϕ^4 theory [35]. The leading order is simply $\hat{g}_0(y) = 1 + y$. The first correction appears at order ϵ^2 and was computed by Fisher and Aharony [4]. We have extended the series, calculating the $O(\epsilon^3)$ term, obtaining

$$\hat{g}_0(y) = 1 + y + \epsilon^2 \frac{N+2}{(N+8)^2} \left\{ 1 + \epsilon \left[1 + \frac{6(3N+14)}{(N+8)^2} \right] \right\} \psi_2(y) + \epsilon^3 \frac{N+2}{(N+8)^2} \psi_3(y) + O(\epsilon^4),$$

(98)

where

$$\psi_2(y) = 2 \int_0^\infty \sqrt{z(1 + \tfrac{1}{4}z)} \ln\left(\sqrt{1 + \tfrac{1}{4}z} + \tfrac{1}{2}\sqrt{z}\right) h(y, z),$$

(99)

$$h(y, z) = -\frac{1}{1+z} + \frac{y}{(1+z)^3} + \frac{1}{2yz}\left(1 + y + z - \sqrt{1 + 2y + 2z + y^2 - 2yz + z^2}\right).$$

We do not report the explicit expression of $\psi_3(y)$ because it is not very illuminating. It can however be obtained from Eqs. (B15), (B16) and (B18) of App. B, where we show how to derive the functions $\psi_2(y)$ and $\psi_3(y)$ from the $O(1/N)$ calculation of $\hat{g}_0(y)$ in d dimensions.

Setting

$$c_i = \epsilon^2 \frac{N+2}{(N+8)^2} \hat{c}_i,$$

(100)

one finds

$$\hat{c}_2 = -0.00752024... \times \left[1 + \epsilon\left(\frac{6(3N+14)}{(N+8)^2} - 0.249301...\right) + O(\epsilon^2)\right],$$

(101)

$$\hat{c}_3 = 0.000191931... \times \left[1 + \epsilon\left(\frac{6(3N+14)}{(N+8)^2} - 0.130607...\right) + O(\epsilon^2)\right],$$

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$$\widehat{c}_4 = -0.0000081420... \times \left[1 + \epsilon\left(\frac{6(3N+14)}{(N+8)^2} - 0.003053...\right) + O(\epsilon^2)\right],$$

$$\widehat{c}_5 = 0.0000004391... \times \left[1 + \epsilon\left(\frac{6(3N+14)}{(N+8)^2} + 0.117278...\right) + O(\epsilon^2)\right],$$

etc....

One also obtains

$$S_M = 1 + \epsilon^2 \frac{N+2}{(N+8)^2}\left[1 + \frac{6(3N+14)}{(N+8)^2}\epsilon\right]\psi_2(-1) + \epsilon^3 \frac{N+2}{(N+8)^2}\psi_3(-1) + O(\epsilon^4), \qquad (102)$$

where

$$\psi_2(-1) = -0.00772078..., \qquad \psi_3(-1) = 0.00189984..., \qquad (103)$$

and

$$S_Z = 1 + \epsilon^2 \frac{N+2}{(N+8)^2}\left[1 + \frac{6(3N+14)}{(N+8)^2}\epsilon\right]\psi_2'(-1) + \epsilon^3 \frac{N+2}{(N+8)^2}\psi_3'(-1) + O(\epsilon^4), \qquad (104)$$

where

$$\psi_2'(-1) = 0.0156512..., \qquad \psi_3'(-1) = -0.0038246.... \qquad (105)$$

In order to get quantitative estimates from the perturbative ϵ-expansion, one should first resum the series and then evaluate the resulting expression at $\epsilon = 1$. Usually resummations are performed assuming the Borel summability of the ϵ-series. As in the case of the g-expansion, a considerable improvement is obtained if one uses the knowledge of the singularity of the Borel transform [18], $b_s = -(N+8)/3$. We have used the procedure described in the previous subsection. Again, since the ϵ-series of c_i, $S_M - 1$ and $S_Z - 1$, have the form $R(\epsilon) = \epsilon^2 \sum_{k=0} a_i \epsilon^i$, we applied the resummation method to $R(\epsilon)$ and to $R(\epsilon)/\epsilon^2$. In Table V we present results for both choices. Since we use a series with only two terms the estimates are not very precise as the large difference between the results obtained with the two methods indicates.

One can also try to get estimates for two-dimensional $O(N)$ models, i.e. for $\epsilon = 2$. By resumming the series of $c_2(\epsilon)$ and $S_M(\epsilon)$, we find: $c_2 = -0.0010$ and $S_M = 0.9989$ for $N = 1$, which must be compared with the exact results [36] $c_2 = -0.000793...$ and $S_M = 0.999196...$; $c_2 = -0.0013$ and $S_M = 0.9987$ for $N = 3$, to be compared with the strong-coupling results [37] $c_2 = -0.0012(2)$ and $S_M = 0.9987(2)$. In both cases the agreement is satisfactory. Instead, when resumming the series divided by ϵ^2 the agreement is poorer. We find $c_2 = -0.0026$ and $S_M = 0.9973$ for $N = 1$ and $c_2 = -0.0028$ and $S_M = 0.9971$ for $N = 3$. A posteriori, it thus appears that the estimates obtained from the resummation of the complete series $R(\epsilon)$ are more reliable. This is confirmed by the three-dimensional analysis where the estimates obtained by considering $R(\epsilon)$ are those which are in better agreement with the strong-coupling and g-expansion estimates.

For quantities which are exactly known in two dimensions, one can modify the ϵ-series to obtain a new expansion which gives the exact value for $\epsilon = 2$. This can achieved [38] by defining for a generic observable R, with ϵ-series $R(\epsilon)$,

$$\bar{R}(\epsilon) = \frac{R(\epsilon) - R(\epsilon = 2)}{2 - \epsilon} \tag{106}$$

and then rewriting

$$\widehat{R}(\epsilon) = R^{exact} + (2 - \epsilon)\bar{R}(\epsilon), \tag{107}$$

where R^{exact} is the exactly known value of R in two dimensions. In other words one subtracts to the original series its value for $\epsilon = 2$ and then sums the exact two-dimensional result. Estimates for $\epsilon = 1$ are obtained by resumming the perturbative expansion of $\bar{R}(\epsilon)$ and computing $\widehat{R}(1)$. We applied the method to the Ising model, i.e. $N = 1$, since in this case, the coefficients c_i and S_M are exactly known [36]. As before, one can also apply the same procedure to $R(\epsilon)/\epsilon^2$, defining

$$\bar{R}_2(\epsilon) = \frac{1}{2 - \epsilon}\left[\frac{R(\epsilon)}{\epsilon^2} - \frac{R(\epsilon = 2)}{4}\right] \tag{108}$$

and then writing

$$\widehat{R}_2(\epsilon) = \frac{\epsilon^2}{4}R^{exact} + \epsilon^2(2 - \epsilon)\bar{R}_2(\epsilon). \tag{109}$$

From a conceptual point of view, Eq. (109) appears preferable to Eq. (107). Indeed Eq. (107) gives the exact value for $d = 2$, but it does not reproduce the correct value for $d = 4$. Eq. (109) instead predicts correctly $\widehat{R}_2(\epsilon) \sim O(\epsilon^2)$ for $\epsilon \to 0$. In any case we report the results obtained with both methods in Table V. They are referred to as "improved"-ϵ expansion. The estimates are in good agreement with the other results. Notice also that the large discrepancy between the two different resummations of the unconstrained ϵ-expansion is here significantly reduced.

E. Strong-coupling analysis

In this subsection we evaluate some of the quantities introduced in Sec. III A by analyzing the strong-coupling expansion of the two-point function $G(x)$ in lattice O(N) non-linear σ models. We recall that non-linear σ models and ϕ^4 models possessing the same internal symmetry O(N) describe the same critical behavior, thus giving rise to the same universal two-point function $\widehat{g}_0(y)$ in the critical limit $M_G \to 0$.

By employing a character-like approach [39], we have calculated the strong-coupling expansion of $G(x)$ up to 15th order on the cubic lattice and 21st order on the diamond lattice for the corresponding nearest-neighbor formulations. In App. D we present the 15th order strong-coupling expansion of $G(x)$ on the cubic lattice. In App. E we report the 21st order strong-coupling series of the magnetic susceptibility and of the second moment of $G(x)$ on the diamond lattice for $N = 1, 2, 3$.

We mention that longer strong-coupling series, up to 21st order, of the lowest moments of $G(x)$ on the cubic and b.c.c. lattices have been recently calculated by a linked-cluster expansion technique, and an updated analysis of the critical exponents γ and ν has been presented [40]. For $N = 0$ even longer series have been calculated for various lattices [41–43].

In our strong-coupling analysis, we took special care in devising improved estimators for the physical quantities c_i and S_M, because better estimators can greatly improve the stability of the extrapolation to the critical point. Our search for optimal estimators was guided by the large-N limit of lattice $O(N)$ σ models.

In the large-N limit of $O(N)$ σ models on the cubic lattice the following exact relations hold in the high-temperature phase, i.e. for $\beta < \beta_c$,

$$\hat{u}_2^\infty(M_G) \equiv \hat{u}_2 = -\frac{1}{20} M_G^2,$$

$$\hat{u}_3^\infty(M_G) \equiv \hat{u}_3 = \frac{1}{840} M_G^4, \tag{110}$$

etc..., which vanish for $T \to T_c$, i.e. for $M_G^2 \to 0$, leading to the expected result $c_i = 0$. Similarly on the diamond lattice one obtains

$$\hat{u}_2^\infty(M_G) \equiv \hat{u}_2 = -\frac{1}{20} M_G^2,$$

$$\hat{u}_3^\infty(M_G) \equiv \hat{u}_3 = \frac{1}{7560} M_G^4 \frac{1 + \frac{3}{4} M_G^2}{1 + \frac{1}{12} M_G^2}, \tag{111}$$

etc... The above formulae can be derived from the large-N limit of the two-point function on the cubic and diamond lattice given in App. A.

We introduce the following quantities

$$\bar{u}_i \equiv \hat{u}_i - \hat{u}_i^\infty(M_G), \tag{112}$$

whose limits for $T \to T_c$ are still c_i. At $N = \infty$ $\bar{u}_i$ are optimal estimators of c_i, since

$$\bar{u}_i(\beta) = \bar{u}_i^* = c_i = 0 \tag{113}$$

for $\beta < \beta_c$, i.e. off-critical corrections are absent. It turns out that the use of $\bar{u}_i$, beside improving the estimates for large values of N, leads also to more precise estimates of c_i at low values of N. Strong-coupling series of $\bar{u}_i$ can be easily obtained from the strong-coupling expansion of $G(x)$.

On the lattice, in the absence of a strict rotational invariance, one may actually define different estimators of the mass-gap having the same critical limit. On the cubic lattice one may consider μ obtained by the long-distance behavior of the side wall-wall correlation constructed with $G(x)$, or equivalently the solution of the equation

$$\widetilde{G}^{-1}(i\mu, 0, 0) = 0. \tag{114}$$

In view of a strong-coupling analysis, it is convenient to use another estimator of the mass-gap derived from μ [44,6]:

$$M_c^2 = 2\left(\cosh\mu - 1\right), \tag{115}$$

which has an ordinary strong-coupling expansion (μ has a singular strong-coupling expansion, starting with $-\ln\beta$). One can easily check that $M_c/\mu \to 1$ in the critical limit. A

similar quantity M_d^2 can be defined on the diamond lattice, as shown in App. A 3 (cfr. Eq. (A25)). One may then consider the dimensionless ratios M_c^2/M_G^2 and M_d^2/M_G^2 respectively on the cubic and diamond lattices, and evaluate their fixed-point limit S_M, which by universality must be the same.

In order to determine the coefficients c_2 and c_3 of the low-momentum expansion of $\widehat{g}_0(y)$ and the mass-ratio S_M, we analyzed the strong-coupling series of $\bar{u}_2$ and $\bar{u}_3$ (defined in Eq. (112)), and those of the ratios M_c^2/M_G^2 and M_d^2/M_G^2 respectively on the cubic and diamond lattice.

On the cubic lattice the available series of $\bar{u}_2$, $\bar{u}_3$ and $M_c^2/M_G^2 - 1$ are respectively of the form $\beta^4 \sum_{i=0}^9 a_i\beta^i$, $\beta^3 \sum_{i=0}^9 a_i\beta^i$, and $\beta^6 \sum_{i=0}^5 a_i\beta^i$; except for $N = 1$ where they are of the form $\beta^6 \sum_{i=0}^7 a_i\beta^i$, $\beta^5 \sum_{i=0}^7 a_i\beta^i$, and $\beta^8 \sum_{i=0}^3 a_i\beta^i$. These series can be derived from the strong-coupling expansion of $G(x)$ presented in App. D. On the diamond lattice the available series of $\bar{u}_2$, $\bar{u}_3$ and $M_d^2/M_G^2 - 1$ are respectively of the form $\beta^6 \sum_{i=0}^{13} a_i\beta^i$, $\beta^5 \sum_{i=0}^{13} a_i\beta^i$, and $\beta^6 \sum_{i=0}^9 a_i\beta^i$; except for $N = 1$ where they are of the form $\beta^8 \sum_{i=0}^{11} a_i\beta^i$, $\beta^7 \sum_{i=0}^{11} a_i\beta^i$, and $\beta^8 \sum_{i=0}^7 a_i\beta^i$

We constructed various types of approximants to the above series, such as Padé approximants (PA's), Dlog-Padé approximants (DPA's) and first-order inhomogeneous integral approximants (IA's) [45]. We then evaluated them at the critical point β_c in order to obtain an estimate of the corresponding fixed point value. For the cubic lattice and most values of N, β_c is available in the literature from strong-coupling and numerical Monte Carlo studies (see for example Refs. [23,40,42,46–49]). When β_c was not known (as in the case of diamond lattice models for $N > 0$), we estimated it by performing a IA analysis of the strong-coupling series of the magnetic susceptibility $\chi = \sum_x G(x)$. In our analysis errors due to the uncertainty on the value of β_c turned out negligible. The values of β_c used in our calculations are reported in Table II.

In the analysis of a series such as $A = \beta^m \sum_{i=0}^n a_i\beta^i$, we constructed approximants to the nth order series $\beta^{-m}A = \sum_{i=0}^n a_i\beta^i$, and then derived the original quantity from them. Given a nth order series, we considered the following quasi-diagonal approximants:

(i) $[l/m]$ PA's with

$$l + m \geq n - 2,$$
$$l, m \geq \frac{n}{2} - 2; \tag{116}$$

(ii) $[l/m]$ DPA's with

$$l + m + 1 \geq n - 2,$$
$$l, m \geq \frac{n}{2} - 2; \tag{117}$$

(iii) $[m/l/k]$ IA's with

$$m + l + k + 2 = n,$$
$$\lfloor (n-2)/3 \rfloor - 1 \leq m, l, k \leq \lceil (n-2)/3 \rceil + 1. \tag{118}$$

In the case (i) and (ii), l, m are the orders of the polynomials respectively in the numerator and denominator of the PA of the series at hand, or of the series of the logarithmic derivative

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in the case of DPA. In the case of integral approximants, m, l, k are the orders of the polynomial Q_m, P_l and R_k defined by the first-order linear differential equation

$$Q_m(x)f'(x) + P_l(x)f(x) + R_k(x) = O\left(x^{k+l+m+2}\right),\tag{119}$$

whose solution provides an approximant of the series at hand.

In Table III we show some details of the above-described analysis for the series on the cubic lattice and for selected values of N, including those physically interesting. We report there various estimates as obtained from the corresponding plain series, and the resummations by PA's, DPA's and IA's. On the cubic lattice, for $N \leq 8$, the plain series of $\bar{u}_2$ already provides a good estimate of c_2, indeed the values at β_c of the series using i terms appear to oscillate around an approximately constant value. In these cases as an estimate of c_2 we can take the average of the values of $\bar{u}_2$ at β_c obtained using n and $n-1$ terms (the errors reported in Table III are related to their difference). For the c_2 corresponding to other values of N and for c_3 and S_M, such oscillations are not observed but the results from the plain series are close to their resummations. In these cases in Table III we report just the value at β_c using all available terms of the series. From the PA, DPA, IA analysis we take the average of the values at β_c of the non-defective approximants using all available terms of the series. As an indicative error from each analysis we consider the square root of the variance around the estimate of the results from all non-defective approximants specified above. PA's, DPA's and IA's are considered defective when they present spurious singularities close to the real axis for $\text{Re}\beta \lesssim \beta_c$. More precisely we considered defective the approximants with spurious singularities located in the region $0 \leq \text{Re}\beta \leq 1.1 \times \beta_c$ (sometimes $0 \leq \text{Re}\beta \leq 1.2 \times \beta_c$) and $|\text{Im}\beta| \leq 0.5 \times \beta_c$. Anyway, our final results turned out to be quite insensitive to this choice. Most of the approximants we considered turned out non-defective. Similarly in Table IV we report results of the analysis of the series on the diamond lattice. In this case we do not report estimates from the plain series because they appear not to be reliable and very far from their resummations.

Table V summarizes our calculations. The final estimates of c_2, c_3 and S_M reported in Table V synthetize the results from all the analyses we performed, and the reported errors are a rough estimate of the uncertainty. Final results are rather accurate taking into account the smallness of the effect we are looking at. Universality among the cubic and diamond lattices is in all cases well verified and gives further support to our final estimates. Our results are in good agreement with the estimates obtained from the g- and $1/N$ expansion. Only for c_2 and small values of N one notices a small discrepancy, probably due to confluent singularities, which are not properly handled by the approximants we considered. We also tried different resummation methods [50] which take into account the subleading corrections. However, in this case, most of the approximants turned out to be defective and no reliable estimate could be obtained.

Our strong-coupling analysis represents a substantial improvement with respect to earlier results reported in Ref. [6] for the Ising model, and obtained by an analysis of the strong-coupling series calculated in Refs. [1,51]: $c_2 = -5.5(1.5) \times 10^{-4}$, $c_3 = 0.05(2) \times 10^{-4}$ on the cubic lattice, and $c_2 = -7.1(1.5) \times 10^{-4}$ and $c_3 = 0.09(3) \times 10^{-4}$ on the b.c.c. lattice. Other strong-coupling results can be found in Ref. [44]. Our analysis achieves a considerable improvement with respect to such earlier works essentially for two reasons: we use longer

series and improved estimators, see Eq. (112), which allow a more stable extrapolation to the critical limit. Estimates from the analysis of the strong-coupling series of the standard variables $\hat{u}_i$, defined in Eq. (59), are much less precise, although consistent with those obtained from $\bar{u}_i$.

F. Summary

We have studied the low-momentum behavior of the two-point function in the critical limit by considering several approaches: $1/N$-expansion, g-expansion, ϵ-expansion and strong-coupling expansion. A summary of our results can be found in Table V.

From the analysis of our strong-coupling series we have obtained quite accurate estimates of the coefficients c_2, c_3 of the low momentum expansion (54). Asymptotic large-N formulae (70) and (72) are clearly approached by our strong-coupling results, but only at rather large values of N. The same behavior was already observed for other quantities like critical exponents [2] and the zero-momentum renormalized four-point coupling [23]. We have also computed the universal function $\hat{g}_0(y)$ in the g-expansion in fixed dimension to order $O(g^4)$ and in ϵ-expansion to order $O(\epsilon^3)$. The corresponding estimates of c_2, c_3 and S_M are in reasonable agreement with the strong-coupling results.

For all values of N the coefficients c_2 and c_3 turn out very small and the pattern (71) is verified. Furthermore relation (76) is satisfied within the precision of our analysis.

The few existing Monte Carlo results for the low-momentum behavior of the two-point Green's function are consistent with our determinations but are by far less precise. Using Refs. [52–54] one estimates $c_2 = -13(17) \times 10^{-4}$ for self-avoiding walks, which correspond to $N = 0$. In Ref. [55] the authors give a bound on $\sqrt{S_M}$ for the Ising model ($N = 1$), from which $-1.2 \times 10^{-3} < S_M - 1 < 0$, which must be compared with our estimate $S_M - 1 = -2.5(5) \times 10^{-4}$. Monte Carlo simulations of the XY model ($N = 2$) shows that $S_M \simeq 1$ within 0.1% [48], which is consistent with our strong-coupling result $S_M - 1 \simeq -3.5(5) \times 10^{-4}$.

We can conclude that in the critical region of the symmetric phase the two-point Green's function for all N from zero to infinity is almost Gaussian in a large region around $k^2 = 0$, i.e., $|k^2/M_G^2| \lesssim 1$. The small corrections to Gaussian behavior are dominated by the $(k^2)^2$ term in the expansion of the inverse propagator. Via the relation (1) such low-momentum behavior could be probed by scattering esperiments by observing the low-angle variation of intensity. A similar low-momentum behavior of the two-point correlation function has been found in two-dimensional O(N) models [36,37,56]. Substantial differences from Gaussian behavior appear at sufficiently large momenta, where $\widetilde{G}(k)$ behaves as $1/k^{2-\eta}$ with $\eta \neq 0$ (although η is rather small: $\eta \simeq 0.03$ for $0 \leq N \leq 3$).

IV. ANISOTROPY OF $G(x)$ AT LOW MOMENTUM AND IN THE CRITICAL REGIME

In this Section we will study anisotropic effects on the two-point function due to the lattice structure. We will mainly consider three-dimensional lattices with cubic symmetry. However, whenever possible, we will give expressions for general d-dimensional lattices with hypercubic symmetry, so that one can recover the results for the square lattice and compare

with perturbative series in $d = 4 - \epsilon$. We will also comment briefly and present some results for the triangular, honeycomb and diamond lattices.

A. Notations

In the following subsections we will compute the exponent $\rho = 2 + \eta - \eta_4$ defined in Eq. (35). It can be derived directly from Eq. (31) or Eq. (34) or by studying the weighted moments $\overline{q}_{4,j} = q_{4,j}/m_0$ where $q_{4,j}$ is defined in Eq. (39) and $m_0 \equiv \chi$. Indeed for $M_G \to 0$,

$$\overline{q}_{4,j} \sim M_G^{-4-2m+\rho}. \tag{120}$$

We will also compute the universal function $\hat{g}_4(y)$. In particular we will be interested in the first terms of its expansion in powers of y around $y = 0$:

$$\hat{g}_4(y) = 1 + \sum_{i=1} d_i y^i, \tag{121}$$

where $d_i \equiv c_{4,i}$ (cfr. Eq. (30)). The coefficients d_i can be easily obtained from the expressions of the moments $q_{4,m}$. For $M_G \to 0$, we find

$$\frac{\overline{q}_{4,1}}{\overline{q}_{4,0}} \longrightarrow 4(d+8)\left(1 - \tfrac{1}{2}d_1\right) M_G^{-2},$$

$$\frac{\overline{q}_{4,2}}{\overline{q}_{4,0}} \longrightarrow 24(d+8)(d+10)\left(1 - \tfrac{2}{3}d_1 - \tfrac{2}{3}c_2 + \tfrac{1}{3}d_2\right) M_G^{-4}, \tag{122}$$

and so on. From (122) it is easy to derive expressions for $r_i \equiv u_{4,i}/u_{4,0}$ whose critical limit is d_i. In particular

$$r_1 = 2 - \frac{M_G^2}{2(d+8)}\frac{\overline{q}_{4,1}}{\overline{q}_{4,0}}. \tag{123}$$

B. Breaking of rotational invariance in the large-N limit

In the large-N limit lattice $O(N)$ models become massive Gaussian theories which can be solved exactly. If one considers theories defined on Bravais lattices one has in the large-N limit

$$\tilde{G}^{-1}(k) = c\beta\,(\overline{k}^2 + M_G^2), \tag{124}$$

where $\overline{k}^2$ is defined by Eq. (8). The relation between M_G^2 and β is given by the gap equation. The constant c is lattice-dependent and will not play any role in the discussion. Specific examples are the nearest-neighbor Hamiltonians of the form (5) for which we have collected in Appendix A general expressions for various three-dimensional lattices, the cubic, diamond and f.c.c. lattice. Analogous formulae for some two-dimensional lattices, the square, triangular, and honeycomb lattice can be found in Ref. [56]. The function $\overline{k}^2$ has the

properties mentioned at the beginning of Sect. II A and a multipole expansion of the type (10) for lattices with cubic symmetry. For other Bravais lattices the only difference is the presence of different multipole combinations. Considering first lattices with (hyper)-cubic symmetry, from Eqs. (10) and (18), we find for $M_G \to 0$

$$\tilde{G}^{-1}(k) = c\beta M_G^2 \left[1 + y + M_G^2(e_{2,0}y^2 + e_{4,0}Q_4(k/M_G))\dots \right]. \tag{125}$$

Comparing with Eqs. (24) and (34) we get immediately $\rho = 2$ and $\hat{g}_4(y) = 1$, i.e. $d_i = 0$ for all $i \neq 0$.

In the large-N limit one can easily verify the universality properties of the ratios defined in Eq. (40). Indeed for generic Hamiltonians in the critical limit $M_G \to 0$ (keeping the dimension of the lattice d generic) we have

$$\overline{m}_{2m} \longrightarrow 2^m m! \left(\prod_{i=0}^{m-1} (d + 2i) \right) M_G^{-2m}, \tag{126}$$

$$\frac{\overline{q}_{4,m}}{\overline{q}_{4,0}} \longrightarrow 2^m (m+1)! \left(\prod_{i=0}^{m-1} (d + 8 + 2i) \right) M_G^{-2m}, \tag{127}$$

and

$$\overline{q}_{4,0} \longrightarrow -e_{4,0} \frac{24d(d-1)}{d+2} M_G^{-2}. \tag{128}$$

Notice that the only dependence on the specific Hamiltonian is in the expression of $\overline{q}_{4,0}$. (Exact expressions for some of these quantities are reported for the theory with nearest-neighbor interactions on the cubic, diamond and f.c.c. lattice in Table VI and on the square lattice in Table VII). Universality is then a straightforward consequence of the independence of the ratio (127) from $e_{4,0}$. It should also be noticed that

$$A_{4,m} \equiv \frac{\overline{q}_{4,m}}{\overline{m}_{4+2m}} \sim M_G^2. \tag{129}$$

This shows that, as expected, anisotropic moments are suppressed by two powers of M_G in agreement with the prediction $\rho = 2$. We stress that the universality of $R_{4,m,n}$ is due the fact that there is only one leading irrelevant operator breaking rotational invariance.

It is interesting to notice that such a universality does not hold for moments $\overline{q}_{6,m}$ (or for $\overline{q}_{2l,m}$ for higher values of l) because of the mixings we have mentioned in Sec. II A. For $\overline{q}_{6,m}$ we have for $T \to T_c$

$$\frac{\overline{q}_{6,m}}{\overline{q}_{6,0}} \longrightarrow 2^m (m+1)! \left(1 + \frac{e_{4,0}^2}{e_{6,0}} \frac{8m}{d+12} \right) \left(\prod_{i=0}^{m-1} (d + 12 + 2i) \right) M_G^{-2m}, \tag{130}$$

which depends on $e_{6,0}$ and $e_{4,0}^2$, a consequence of the fact that $Q_4(k)^2$ contains a term of the form $k^2 Q_6(k)$. Thus ratios of the form (40) built with $\overline{q}_{6,m}$ are not universal.

Let us now consider the diamond lattice. In this case not only rotational invariance is broken, but also the parity symmetry. As the leading anisotropic operator is $O_{4,0}(x)$ the behaviour of the leading anisotropic corrections is identical to that we have discussed above.

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Therefore $\rho = 2$ also in this case. Moreover the invariant ratios $R_{4,m,n}$ are identical for the diamond lattice and for the other Bravais lattices with cubic symmetry. Eq. (127) is exact for the diamond lattice as well.

To discuss parity-breaking effects we must consider odd moments of $G(x)$. In particular one finds that, for $M_G \to 0$,

$$\bar{q}_{3,0} \equiv \frac{q_{3,0}}{m_0} \longrightarrow \frac{1}{6\sqrt{3}}, \tag{131}$$

where

$$q_{3,0} \equiv \sum xyz \, G(x,y,z). \tag{132}$$

Thus parity-breaking effects vanish as $M_{G,}^3$, i.e. $\rho_p = 3$, faster than the anisotropic effects we have considered previously.

Finally let us consider lattices which do not have cubic invariance, such as the triangular and the honeycomb one. In Table VII we report the large-N limit of some of the lowest spherical and non-spherical moments of $G(x)$ for the models with nearest-neighbor interactions.

For the triangular lattice one should consider the multipole expansion (42). In this case the leading term breaking rotational invariance is proportional to $T_6(k)$ and thus we have $\rho = 4$. This is indeed confirmed by the fact that, for $M_G \to 0$,

$$B_{6,m} \equiv \frac{\bar{t}_{6,m}}{\overline{m}_{6+2m}} \sim M_G^4, \tag{133}$$

where $\bar{t}_{6,m} = t_{6,m}/m_0$ and $t_{6,m}$ is defined in Eq. (43). As in the cubic case, it is immediate to verify the universality of ratios of the form given in Eq. (40) with $t_{6,m}$ instead of $q_{4,m}$, which is a consequence of the uniqueness of the leading operator breaking rotational invariance. Universality follows from the fact that, for $T \to T_c$,

$$\frac{\bar{t}_{6,m}}{\bar{t}_{6,0}} \longrightarrow \frac{2^{2m}(m+1)!(m+5)!}{5!} M_G^{-2m}, \tag{134}$$

independently of the specific Hamiltonian.

For the honeycomb lattice one must also consider the breaking of parity. Considering the odd moment

$$t_{3,0} = \sum (x^3 - 3y^2 x)G(x,y), \tag{135}$$

(cfr. Eq. (49)) one finds

$$\bar{t}_{3,0} \equiv \frac{t_{3,0}}{m_0} \longrightarrow \frac{1}{2}. \tag{136}$$

Thus, as in the diamond case, parity breaking effects vanish as M_G^3, i.e. $\rho_p = 3$.

C. Analysis to order $1/N$

In the previous subsection we computed the exponent ρ for $N \to \infty$ for lattices with cubic symmetry, finding $\rho = 2$. Now we want to compute the $1/N$ corrections, i.e. the value of $\sigma \equiv \sigma_4 = \eta - \eta_4$ (cfr. Eq. (35)), which is the anomalous dimension of the operator $O_{4,0}(x)$. More generally we can compute the exponents η_{2l} defined in Eq. (31) for arbitrary l. Notice that in this way we will also obtain the $1/N$ correction to ρ for the triangular lattice which depends on η_6.

In d dimensions, we consider the following representation of the inverse two-point function where the $O(1/N)$ correction has been included

$$\beta^{-1}\widetilde{G}^{-1}(k) = \overline{k}^2 + \beta^{-1}Z_G^{-1}M_G^2 + \frac{1}{N}\int \frac{d^d p}{(2\pi)^d}\Delta(p)\left(\frac{1}{\overline{p+k}^2 + M_G^2} - \frac{1}{\overline{p}^2 + M_G^2}\right). \tag{137}$$

Here $\overline{k}^2$ is the inverse lattice propagator defined in Eq. (8),

$$Z_G^{-1} = \frac{1}{2}\frac{\partial^2 \widetilde{G}(k)^{-1}}{\partial k_\mu^2}\bigg|_{k_\mu = 0} = \frac{Z_0}{M_G^2}, \tag{138}$$

and

$$\Delta^{-1}(p) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\frac{1}{\left(\overline{q+p}^2 + M_G^2\right)\left(\overline{q}^2 + M_G^2\right)}. \tag{139}$$

The following statements can be checked explicitly in Eq. (137) and hold to all orders of the $1/N$ expansion: (i) in the limit $M_G \to 0$ the function $\widetilde{G}^{-1}(k, M_G)$ is spherically symmetric (i.e. it depends only on $y \equiv k^2/M_G^2$, apart from an overall factor); (ii) the only non-spherically symmetric contribution that may appear in $\widetilde{G}^{-1}(k, M_G)$ to $O(M_G^4)$ can be reduced to a spherically symmetric function multiplied by $Q_4(k)$. These statements are simply a consequence of applying the discrete and continuous symmetry properties to all integrals appearing in the asymptotic expansion in M_G of the relevant Feynman integrals. They prove to all orders in $1/N$ the validity of the expansion (34).

To compute the anomalous dimension η_{2l} to order $1/N$ we will use the trick we explained in Sec. II A. If one considers a particular Hamiltonian such that $g_{2l}(y, M_G) = 0$ for $0 \leq l \leq \bar{l}$, then $\widetilde{G}^{-1}(k)$ has an expansion of the form (34) with $\eta_4 \to \eta_{2\bar{l}}$ and $\hat{g}_4(y) \to \hat{g}_{2\bar{l}}(y)$. In the $1/N$-expansion, to order $1/N$ this can be achieved by considering Hamiltonians such that, for $k \to 0$, (to simplify the notation from now on we write l instead of $\bar{l}$)

$$\overline{k}^2 = k^2 + rk^{2l} + O(k^{2l+2}) \tag{140}$$

where $k^{2l} \equiv \sum_\mu k_\mu^{2l}$. The limit $M_G \to 0$ can then be easily obtained by evaluating massless continuum integrals, and taking the contribution proportional to r, which is the only term relevant to our computation. In this limit we obtain

$$\Delta^{-1}(p) \longrightarrow \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[(q+p)^2 + r(q+p)^{2l}][q^2 + rq^{2l}]}$$

$$\approx \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q+p)^2} - r \int \frac{d^d q}{(2\pi)^d} \frac{q^{2l}}{(q^2)^2(p+q)^2}$$

$$\approx \Delta_0^{-1}(p) \left(1 - rB_l \frac{p^{2l}}{p^2}\right), \tag{141}$$

where

$$\Delta_0^{-1}(p) = \frac{1}{2}(p^2)^{\frac{d}{2}-2} \frac{\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2} - 1)^2}{(4\pi)^{\frac{d}{2}}\Gamma(d-2)}, \tag{142}$$

$$B_l = (4 - d) \frac{\Gamma\left(\frac{d}{2} + 2l - 2\right)\Gamma(d-2)}{\Gamma(d + 2l - 3)\Gamma\left(\frac{d}{2} - 1\right)}, \tag{143}$$

and we have discarded rotationally invariant terms proportional to r, since they will not contribute to the final result.

We must now identify the singular contribution in the limiting form of Eq.(137):

$$\beta^{-1}\widetilde{G}^{-1}(k) \longrightarrow k^2 + rk^{2l} + \frac{1}{N} \int \frac{d^d p}{(2\pi)^d} \Delta_0(p) \left[1 - rB_l \frac{p^{2l}}{p^2}\right]^{-1} \left[(p+k)^2 + r(p+k)^{2l}\right]^{-1}$$

$$\approx k^2 + \frac{1}{N} \int \frac{d^d p}{(2\pi)^d} \frac{\Delta_0(p)}{(p+k)^2} + r\left\{k^{2l} + \frac{1}{N} \int \frac{d^d p}{(2\pi)^d} \frac{\Delta_0(p)}{(p+k)^2} \left[B_l \frac{p^{2l}}{p^2} - \frac{(p+k)^{2l}}{(p+k)^2}\right]\right\}$$

$$\approx k^2 \left(1 - \frac{1}{N}\eta_1 \ln k\right) + rk^{2l} \left(1 - \frac{1}{N}\eta_{2l,1} \ln k\right). \tag{144}$$

The coefficients η_1 and $\eta_{2l,1}$ are related to the $1/N$ expansion of the exponents η and η_{2l}:

$$\eta = \frac{\eta_1}{N} + O\left(\frac{1}{N^2}\right), \tag{145}$$

and

$$\eta_{2l} = \frac{\eta_{2l,1}}{N} + O\left(\frac{1}{N^2}\right). \tag{146}$$

By simple manipulations one obtains

$$\eta_1 = -\frac{4\Gamma(d-2)}{\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2} - 2)\Gamma(\frac{d}{2} - 1)\Gamma(\frac{d}{2} + 1)}, \tag{147}$$

and

$$\eta_{2l,1} = \frac{d(d-2)}{(d - 2 + 4l)(d - 4 + 4l)} \left[1 + 2\frac{\Gamma(2l+1)\Gamma(d-2)}{\Gamma(2l + d - 3)}\right] \eta_1. \tag{148}$$

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Notice the following properties:

$$\eta_{2,1} = \eta_1 \qquad \text{for all} \quad d, \tag{149}$$

$$\eta_{2l,1} \to \eta_1 \qquad \text{for} \quad d \to 2, \tag{150}$$

$$\eta_{2l,1} = \frac{3}{4l-1}\eta_1 \qquad \text{for} \quad d = 3, \tag{151}$$

$$\eta_{2l,1} \to \frac{3}{l(2l+1)}\eta_1 \qquad \text{for} \quad d \to 4. \tag{152}$$

Therefore for $d = 3$ we find

$$\eta_1 = \frac{8}{3\pi^2}, \tag{153}$$

which agrees with the known result, and

$$\sigma = \eta - \eta_4 = \frac{32}{21\pi^2 N} + O\left(\frac{1}{N^2}\right). \tag{154}$$

For η also the $O(1/N^2)$ and $O(1/N^3)$ corrections are known [57]. The coefficient σ_1 of the $1/N$ expansion is very small. Thus, at least for N sufficiently large, say $N \gtrsim 8$, where the $1/N$ expansion is known to work reasonably well, corrections to the Gaussian value of ρ are very small. In two dimensions, and to $O(1/N)$, there are no corrections to the Gaussian value because of Eq. (150), i.e. the first coefficient of the expansion of the anomalous dimension is zero to $O(1/N)$. One might only observe (suppressed) logarithmic corrections to canonical scaling for all l. It is easy to check in perturbation theory that this holds exactly for all $N \geq 3$.

Alternatively the exponents η_{2l} could have been computed from Eqs. (27) and (31).

The computation of the universal function $\hat{g}_4(y)$ is particularly involved and is presented in App. C. In this Section we will only give the values of the coefficients d_i which appear in the low-momentum expansion (cfr. Eq. (121)). We found

$$d_1 = -\frac{0.00206468...}{N} + O\left(\frac{1}{N^2}\right), \tag{155}$$

$$d_2 = \frac{0.00007378...}{N} + O\left(\frac{1}{N^2}\right),$$

$$d_3 = -\frac{0.00000424...}{N} + O\left(\frac{1}{N^2}\right),$$

etc....

D. g-expansion analysis

The critical exponent σ and the scaling function $\hat{g}_4(y)$ can also be evaluated in the g-expansion. For this purpose we calculated the one-particle irreducible two-point function $\Gamma_{O_4}(k, M_G)$ defined in Eq. (27). By a three-loop calculation one finds for the bare correlation function

$$\Gamma_{O_4}(k, M_G) = Q_4(k) + g_0^2 \frac{N+2}{6} J_2(k, M_G)$$
$$- g_0^3 \frac{(N+2)(N+8)}{108} \left[J_{3,1}(k, M_G) + 4J_{3,2}(k, M_G) \right] + O(g_0^4) \tag{156}$$

where

$$J_2(k, m) = \int \frac{d^3p}{(2\pi)^3} \frac{Q_4(k-p) A(p,m)}{\left[(k-p)^2 + m^2 \right]^2}, \tag{157}$$

$$J_{3,1}(k, m) = \int \frac{d^3p}{(2\pi)^3} \frac{Q_4(k-p) A(p,m)^2}{\left[(k-p)^2 + m^2 \right]^2}, \tag{158}$$

$$J_{3,2}(k, m) = \int \frac{d^3p}{(2\pi)^3} \frac{A(p,m) A_Q(p,m)}{(k-p)^2 + m^2}, \tag{159}$$

and

$$A(p, m) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\left[q^2 + m^2 \right] \left[(q+p)^2 + m^2 \right]} = \frac{1}{4\pi p} \arctan \frac{p}{2m}, \tag{160}$$

$$A_Q(p, m) = \int \frac{d^3q}{(2\pi)^3} \frac{Q_4(q)}{\left[q^2 + m^2 \right]^2 \left[(q+p)^2 + m^2 \right]}. \tag{161}$$

By appropriately renormalizing $\Gamma_{O_4}(k, M_G)$ at $k = 0$, one derives

$$\overline{Z}_4 = 1 + \bar{g}^2 \frac{N+2}{(N+8)^2} \frac{40\pi^2}{3} Q_4 \left(\partial/\partial k^2 \right) J_2(k, 1)|_{k=0} \tag{162}$$
$$+ \bar{g}^3 \frac{N+2}{(N+8)^2} \frac{80\pi^2}{9} Q_4 \left(\partial/\partial k \right) \left[3J_2(k, 1) - 4\pi J_{3,1}(k, 1) - 16\pi J_{3,2}(k, 1) \right]_{k=0} + O(\bar{g}^4).$$

Defining $Z_{O_4} \equiv \overline{Z}_4/Z_G$ and using (87), we obtain

$$\gamma_{O_4}(\bar{g}) = \beta(\bar{g}) \frac{\partial \ln Z_{O_4}}{\partial \bar{g}} = \bar{g}^2 \frac{5408}{25515} \frac{N+2}{(N+8)^2} \left[1 + \bar{g} \times 0.045007 + O(\bar{g}^2) \right]. \tag{163}$$

The critical exponent σ is obtained by evaluating $\gamma_{O_4}(g)$ at the fixed-point value of the coupling, i.e.

$$\sigma = \gamma_{O_4}(g^*). \tag{164}$$

The scaling function $\widehat{g}_4(y)$ is obtained from the zero-momentum renormalized function $\Gamma_{O_4,R}$. Indeed setting

$$\Gamma_{O_4,R} = Q_4(k) f_4(g, y), \tag{165}$$

we have

$$\widehat{g}_4(y) = \lim_{g \to g^*} f_4(g, y). \tag{166}$$

The expansion of $f_4(g, y)$ is

$$f_4(g, y) = 1 + \bar{g}^2 \frac{N+2}{(N+8)^2} \frac{40\pi^2}{3} \left[Q_4\left(\partial/\partial k\right) J_2(k, 1) - Q_4\left(\partial/\partial k\right) J_2(k, 1)|_{k=0} \right] \tag{167}$$

$$+ \bar{g}^3 \frac{N+2}{(N+8)^2} \frac{80\pi^2}{9} \left\{ Q_4\left(\partial/\partial k\right) \left[3J_2(k, 1) - 4\pi J_{3,1}(k, 1) - 16\pi J_{3,2}(k, 1) \right] - (k=0) \right\} + O(\bar{g}^4),$$

By expanding $f_4(g, y)$ in powers of y around $y = 0$, one finds

$$d_i = \bar{g}^2 \frac{N+2}{(N+8)^2} \bar{d}_i, \tag{168}$$

and

$$\bar{d}_1 = -\frac{380}{168399} \left[1 + \bar{g} \times 0.105400 + O(\bar{g}^2) \right], \tag{169}$$
$$\bar{d}_2 = \frac{3076}{19702683} \left[1 - \bar{g} \times 0.355629 + O(\bar{g}^2) \right],$$
$$\bar{d}_3 = -\frac{3112}{253320210} \left[1 - \bar{g} \times 0.696450 + O(\bar{g}^2) \right],$$

etc...

In order to get estimates of σ and of the coefficients d_i from the corresponding series, we have employed the procedure outlined in Section III C. Results for σ are reported in Table VIII, and for d_1 in Table X.

E. An ϵ-expansion analysis

The exponents η_{2l} introduced in Sec. II A can be evaluated in the ϵ-expansion of the corresponding ϕ^4 theory. A rather simple two-loop calculation gives

$$\eta = \frac{1}{2} \frac{N+2}{(N+8)^2} \epsilon^2 + O(\epsilon^3) \tag{170}$$

and

$$\eta_{2l} = \frac{3}{2l(2l+1)} \frac{N+2}{(N+8)^2} \epsilon^2 + O(\epsilon^3). \tag{171}$$

We recall that the exponent η is known to $O(\epsilon^4)$ [2]. For σ one then finds

$$\sigma = \eta - \eta_4 = \frac{7}{20} \frac{N+2}{(N+8)^2} \epsilon^2 + O(\epsilon^3). \tag{172}$$

To compute $\hat{g}_4(y)$, we consider the renormalized two-point one-particle irreducible function with an insertion of the operator $O_4(x)$, see (29). To order $O(\epsilon^2)$ we find

$$\hat{g}_4(y) = 1 + \epsilon^2 \frac{N+2}{(N+8)^2} 8\pi^4 \left[Q_4\left(\partial/\partial k\right) J_s(k, 1) - Q_4\left(\partial/\partial k\right) J_s(k, 1)|_{k=0} \right] + O(\epsilon^3). \tag{173}$$

The function $J_s(k, m)$ is the finite part of the integral

33

$$J(k, m) = \int \frac{d^dq}{(2\pi)^d} \frac{d^dp}{(2\pi)^d} \frac{Q_4(k-p)}{[q^2 + m^2]\,[(q+p)^2 + m^2]\,[(k-p)^2 + m^2]^2}. \tag{174}$$

with the $\overline{MS}$ prescription. The expansion of $\widehat{g}_4(y)$ in powers of y gives

$$d_i = \epsilon^2 \frac{N+2}{(N+8)^2} \widehat{d}_i + O(\epsilon^3) \tag{175}$$

and

$$
\begin{aligned}
\widehat{d}_1 &= -0.00354500..., \\
\widehat{d}_2 &= 0.00011715..., \\
\widehat{d}_3 &= -0.00000599...,
\end{aligned} \tag{176}
$$

etc...

F. A strong-coupling analysis

Anisotropy in the two-point function can be studied at finite N by analyzing the strong-coupling expansion of its lowest non-spherical moments.

In order to compute σ, the correction to the Gaussian value of ρ, we analyze the strong-coupling expansion of the ratio $q_{4,0}/m_2$, which behaves as

$$\frac{q_{4,0}}{m_2} \sim M_G^\sigma \sim (T - T_c)^{\sigma\nu} \tag{177}$$

for $T \to T_c$. We recall that in the $1/N$ expansion $\nu = 1 + O(1/N)$, and for $N = 0, 1, 2, 3$ $\nu \simeq 0.588$, $\nu \simeq 0.630$, $\nu \simeq 0.670$, $\nu \simeq 0.705$ respectively [2]. DPA's and IA's of the available strong-coupling series of the ratio $q_{4,0}/m_2$ on both cubic and diamond lattice turned out not to be sufficiently stable to provide satisfactory estimates of σ at any finite value of N.

A more satisfactory analysis has been obtained by employing the so-called critical point renormalization method (CPRM) [58]. The idea of the CPRM is that from two series $D(x)$ and $E(x)$ which are singular at the same point x_0

$$
\begin{aligned}
D(x) &= \sum_i d_i x^i \sim (x_0 - x)^{-\delta}, \\
E(x) &= \sum_i e_i x^i \sim (x_0 - x)^{-\epsilon},
\end{aligned} \tag{178}
$$

one can construct a new series by

$$F(x) = \sum_i \frac{d_i}{e_i} x^i. \tag{179}$$

The function $F(x)$ is singular at $x = 1$ and for $x \to 1$ behaves as

$$F(x) \sim (1 - x)^{-\phi}. \tag{180}$$

The exponent ϕ is related to δ and ϵ by

$$\phi = 1 + \delta - \epsilon. \tag{181}$$

Therefore the analysis of $F(x)$ provides an unbiased estimate of the difference between the critical exponents of the two functions $D(x)$ and $E(x)$. Moreover the series $F(x)$ may be analyzed by employing biased approximants with a singularity at $x_c = 1$.

By applying the CPRM to the strong-coupling series of $q_{4,0}$ and m_2, one can extract an unbiased estimate of σ, by computing the exponent $\phi = 1 - \sigma\nu$ from the resulting series at the singularity $x_0 = 1$. We analyzed this series by biased IA's, considering those indicated in Eq. (118). The estimates of σ we obtained confirm universality between cubic and diamond lattice, although the analysis on the diamond lattice led in general to less stable results. In Table VIII, for selected values of N, we report our estimates of σ, which are essentially obtained from the analysis on the cubic lattice. In order to derive σ from $\sigma\nu$, which is the quantity derived from the strong-coupling analysis, we have used the values of ν available in the literature. See e.g. [40] for an updated collection of results obtained by various numerical and analytic methods. The errors we report are rough estimates of the uncertainty obtained by considering the spread of all the analyses we performed. The values of σ are very small at all values of N, and at large N, say $N \gtrsim 10$, they are consistent with the corresponding $O(1/N)$ prediction, cfr. Eq. (154). By analyzing separately the strong-coupling series of $q_{4,0}$ and m_2 in the case of σ, and taking the difference of their exponents, one obtains consistent but less precise results.

In order to estimate the first non-trivial coefficient d_1 of the expansion of $\widehat{g}_4(y)$, see Eq. (121), one may consider the quantity r_1 defined in Eq. (123). However, as we did for the analysis of c_i in Sec. III, it is better to consider another quantity $\overline{r}_1$ which is defined so that $\overline{r}_1 = 0$ for $N = \infty$ for all $\beta < \beta_c$. For the cubic lattice

$$\overline{r}_1 = 2 - \frac{q_{4,1}M_G^2}{22q_{4,0}} + \frac{M_G^2}{22}, \tag{182}$$

while for the diamond lattice

$$\overline{r}_1 = 2\frac{1 + \frac{2}{33}M_G^2 + \frac{1}{528}M_G^4}{1 + \frac{1}{12}M_G^2} - \frac{q_{4,1}M_G^2}{22q_{4,0}}. \tag{183}$$

In the critical limit $\overline{r}_1 \to d_1$. On the cubic lattice the available series of $\overline{r}_1$ has the form $\beta^4 \sum_{i=0}^{9} a_i\beta^i$, except for $N = 1$ where it has the form $\beta^6 \sum_{i=0}^{7} a_i\beta^i$. These series can be derived from the strong-coupling expansion of $G(x)$ presented in App. D. On the diamond lattice the available series of $\overline{r}_1$ has the form $\beta^6 \sum_{i=0}^{13} a_i\beta^i$, except for $N = 1$ where it has the form $\beta^8 \sum_{i=0}^{11} a_i\beta^i$. The results of the analysis are reported in Table X. Universality between the cubic and diamond lattice is again substantially verified, although the diamond lattice provides in most cases less precise results. The value of d_1 is very small for all N. At large-N the strong-coupling estimate of d_1 is in good agreement with the large-N prediction (155). The estimates are also in satisfactory agreement with the results obtained from the g-expansion and the ϵ-expansion.

We have also obtained estimates of $\sigma_6 = \eta - \eta_6$, i.e. the anomalous dimension of the irrelevant operator

$$O_6(x) \equiv O_{6,0}(x) = \vec{s}(x) \cdot Q_6(\partial)\vec{s}(x), \tag{184}$$

by analyzing the strong-coupling series of $q_{6,0}$. For the cubic lattice the exponent σ_6 is determined from the critical behavior of the ratio $q_{6,0}/m_2$, since

$$\frac{q_{6,0}}{m_2} \sim M_G^{\sigma_6} \sim (T - T_c)^{\sigma_6\nu}, \tag{185}$$

Notice that Eq. (185) is not valid on the diamond lattice, since here $q_{6,0}$ receives contributions from two operators, from $O_6(x)$ and from the leading irrelevant operator responsible for the parity breaking. Results for various values of N are reported in Table IX. They were obtained by applying the CPRM to the series of $q_{6,0}$ and m_2, and by analyzing the resulting series by biased IA's. Like σ, σ_6 is small for all values of N. At large N our estimates compare well with

$$\sigma_6 = \frac{64}{33\pi^2 N} + O\left(\frac{1}{N^2}\right). \tag{186}$$

Finally we compute ρ_p for the diamond lattice. For a Gaussian theory $\rho_p = 3$ and thus $q_{3,0} \to$ constant for $M_G \to 0$. In general, at finite values of N, we write $\rho_p = 3 + \sigma_p$. The exponent σ_p is determined from the critical behavior of $\overline{q}_{3,0}$

$$\overline{q}_{3,0} \sim M_G^{\sigma_p}. \tag{187}$$

In order to estimate σ_p, we applied the CPRM to the series $q_{3,0}$ and χ. We found $0 \leq \sigma_p \lesssim 0.01$ for all $N \geq 0$.

G. The two-dimensional Ising model

We conclude this section by considering the two-dimensional Ising model, for which we present an argument showing that the anomalous dimension of the irrelevant operators breaking rotational invariance is zero.

Let us consider first the square lattice. In this case, for sufficiently large values of $|x|$ (in units of the lattice spacing) the asymptotic behavior of $G(x)$ on the square lattice can be written in the form [59]

$$G(x) \approx \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} \frac{Z(\beta)}{M^2(\beta) + \widehat{p}^2}, \tag{188}$$

where $\widehat{p}^2 = \sum_\mu 4\sin^2(p_\mu/2)$,

$$Z(\beta) = \left[(1 - z^2)^2 - 4z^2\right]^{1/4} \frac{(1 + z^2)^{1/2}}{z}, \tag{189}$$

and

$$M^2(\beta) = \frac{(1 + z^2)^2}{z(1 - z^2)} - 4, \tag{190}$$

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and we have introduced the auxiliary variable

$$z(\beta) = \tanh\beta. \tag{191}$$

This shows that at large distances the breaking of rotational invariance is identical to that of the massive Gaussian model with nearest-neighbor interactions. Therefore $\rho = 2$ exactly.

This value of ρ is confirmed by a strong-coupling analysis of the moments $q_{4,m}$ using the available 21st order strong-coupling series [36]. In particular, on the square lattice we found

$$\lim_{\beta \to \beta_c} \frac{q_{4,0}}{m_2} = \frac{1}{4} \tag{192}$$

within an uncertainty of $O(10^{-5})$.

A formula analogous to Eq. (188) has been conjectured in Ref. [36] for the Ising model on triangular and honeycomb lattices. Thus, also on these lattices, the pattern of breaking of rotation invariance (and parity in the case of the honeycomb lattice) should be that of the corresponding Gaussian theories, which have been described in Sec. IV B. If the conjecture of Ref. [36] is correct, we have $\rho = 4$ for the triangular lattice and $\rho_p = 3$ for the honeycomb lattice.

Again, an analysis of the strong-coupling expansion of $G(x)$ on the triangular and honeycomb lattices supports convincingly this conjecture.

H. Summary

For lattice models with $O(N)$ symmetry we studied the problem of the recovery of rotational invariance in the critical limit. Anisotropic effects vanish as M_G^ρ, when $M_G \to 0$. The universal critical exponent ρ, which is related to the critical dimension of the leading operator breaking rotational invariance, turns out to be 2 with very small N-dependent corrections for the lattices with cubic symmetry. Notice that this behavior is universal and thus should appear in all physical systems which have cubic symmetry. The reader should note that ρ is different from the exponent ω, which parametrizes the leading correction to scaling and which is related to a different rotationally-invariant irrelevant operator. Models defined on lattices with basis, such as the diamond lattice show also a breaking of the parity symmetry. We find that these effects vanish as $M_G^{\rho_p}$, with $\rho_p \approx 3$ for all values of N.

We have also calculated the universal function $\widehat{g}_4(y)$. For $y \lesssim 1$, we find $\widehat{g}_4(y) \approx 1$ with very small corrections.

In our study we considered several approaches, based on $1/N$, g-, ϵ-, and strong-coupling expansions. All results are in good agreement.

In two dimensions we showed that $\rho = 2$ for the square lattice for all $N \geq 3$ and $N = 1$. We conjecture that this is a general result, valid for all values of N. Similar arguments apply to the triangular (honeycomb) lattice: we conjecture $\rho = 4$ (resp. $\rho_p = 3$) for all N.

ACKNOWLEDGMENTS

APPENDIX A: THE LARGE-N LIMIT ON THE CUBIC, F.C.C. AND DIAMOND LATTICE.

In this appendix we present the large-N limit of the two-point function in $O(N)$ σ model with nearest-neighbor interaction on cubic, f.c.c. and diamond lattices.

1. The cubic lattice

The large-N two-point Green's function on the cubic lattice is [60]

$$G(\vec{x}; \vec{y}) = \frac{1}{\beta} \frac{1}{N_s} \sum_{\vec{k}} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{1}{\widehat{k}^2 + z}, \tag{A1}$$

where

$$\widehat{k}^2 = \sum_i \widehat{k}_i^2,$$

$$\widehat{k}_i^2 = 2(1 - \cos k_i), \tag{A2}$$

and N_s is the number of sites. In Eq. (A1) $z = M_G^2$, where M_G is the zero-momentum mass scale. The lowest spherical and non-spherical moments of $G(x)$ are reported in Table VI.

The relation between β and z is determined by the condition $G(0) = 1$. In the infinite volume limit one has

$$\beta = \int_{-\pi}^{\pi} \frac{d^3 k}{(2\pi)^3} \frac{1}{\widehat{k}^2 + z}. \tag{A3}$$

An expression in terms of elliptic integrals can be found in [61]. The critical point β_c is obtained by setting $z = 0$. One finds [62]

$$\beta_c = \frac{2}{\pi^2} \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6} \right) K \left(2\sqrt{3} - 2\sqrt{2} - 3 + \sqrt{6} \right)^2 = 0.252731..., \tag{A4}$$

where K is the complete elliptic integral of the first kind. The expression for β_c in terms of Γ-functions reported in [63] (and quoted in [64]) is unfortunately wrong.

Concerning tha mass-gap estimator introduced in Eq. (115), it is easy to check that for $N = \infty$ and for $\beta \leq \beta_c$

$$\frac{M_c^2}{M_G^2} = 1, \tag{A5}$$

independently of β.

2. The face-centered cubic lattice

The f.c.c. lattice is a regular lattice, thus the solution of the corresponding Gaussian model does not present difficulties. Its coordination number is $c = 12$ and the volume per site is $v_s = \frac{1}{\sqrt{2}}$ (in unity of a^3, where a is the length of a link).

The sites $\vec{x}$ of a finite periodic f.c.c. lattice can be represented in cartesian coordinates by

$$
\begin{aligned}
\vec{x} &= l_1 \vec{\eta}_1 + l_2 \vec{\eta}_2 + l_3 \vec{\eta}_3, \\
l_i &= 1, ... L_i, \\
\vec{\eta}_1 &= \tfrac{1}{\sqrt{2}} \left(0, 1, 1\right), \\
\vec{\eta}_2 &= \tfrac{1}{\sqrt{2}} \left(1, 0, 1\right), \\
\vec{\eta}_3 &= \tfrac{1}{\sqrt{2}} \left(1, 1, 0\right).
\end{aligned}
\tag{A6}
$$

It is easy to derive the massive Gaussian propagator and therefore the large-N two-point Green's function, which is

$$
G(\vec{x}; \vec{y}) = \frac{1}{2\beta} \frac{1}{N_s} \sum_{\vec{k}} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{1}{\Delta(k) + z},
\tag{A7}
$$

where

$$
\Delta(k) = 2 \left(3 - \cos \tfrac{k_1}{\sqrt{2}} \cos \tfrac{k_2}{\sqrt{2}} - \cos \tfrac{k_2}{\sqrt{2}} \cos \tfrac{k_3}{\sqrt{2}} - \cos \tfrac{k_3}{\sqrt{2}} \cos \tfrac{k_1}{\sqrt{2}} \right),
\tag{A8}
$$

and $N_s = L_1 L_2 L_3$ is the number of sites. The momenta $\vec{k}$ run over the reciprocal lattice:

$$
\begin{aligned}
\vec{k} &= \frac{2\pi}{L_1} m_1 \vec{\rho}_1 + \frac{2\pi}{L_2} m_2 \vec{\rho}_2 + \frac{2\pi}{L_3} m_3 \vec{\rho}_3, \\
m_i &= 1, ... L_i, \\
\vec{\rho}_1 &= \tfrac{1}{\sqrt{2}} \left(-1, 1, 1\right), \\
\vec{\rho}_2 &= \tfrac{1}{\sqrt{2}} \left(1, -1, 1\right), \\
\vec{\rho}_3 &= \tfrac{1}{\sqrt{2}} \left(1, 1, -1\right).
\end{aligned}
\tag{A9}
$$

Again $z = M_G^2$. The lowest moments derived from Eq. (A7) are reported in Table VI.

The relation between β and z is determined by the condition $G(0) = 1$. In the infinite volume limit one obtains

$$
\beta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^3 q}{(2\pi)^3} \frac{1}{2D(q) + z},
\tag{A10}
$$

where

$$
D(q) = \sin^2 \tfrac{q_1}{2} + \sin^2 \tfrac{q_2}{2} + \sin^2 \tfrac{q_3}{2} + \sin^2 \tfrac{q_1 - q_2}{2} + \sin^2 \tfrac{q_2 - q_3}{2} + \sin^2 \tfrac{q_3 - q_1}{2}.
\tag{A11}
$$

By setting $z = 0$ one finds [62]

$$\beta_c = \frac{3\Gamma\left(\frac{1}{3}\right)}{2^{20/3}\pi^4} = 0.112055....\tag{A12}$$

It is worth noticing that the function $e_0(k^2)$, defined in Eq. (10), turns out to be the same for cubic and f.c.c. lattices, as a consequence of some trivial symmetries of the angular integration. Therefore the first few spherical moments are equal as functions of M_G^2 even off-criticality, as shown in Table VI.

3. The diamond lattice

The diamond lattice has coordination number $c = 4$. It is not a regular lattice, because not all the sites are related by a translation. It consists of two interpenetrating f.c.c. lattices, and can be regarded as a f.c.c. lattice with a two-point basis. The absence of translation invariance causes a few subtleties in the analysis of models defined on it. One cannot define a Fourier transform which diagonalizes the corresponding Gaussian propagator. Neverthless, observing that sites at even distance in the number of lattice links form regular f.c.c. lattices, one can define a Fourier-like transformation that partially diagonalizes the Gaussian propagator (up to 2×2 matrices).

Setting the lattice spacing (i.e. the length of a link) $a = 1$, the sites $\vec{x}$ of a finite periodic diamond lattice can be represented in cartesian coordinates by

$$\begin{aligned}
\vec{x} &= \vec{x}\,' + p\,\vec{\eta}_p \\
\vec{x}\,' &= l_1\,\vec{\eta}_1 + l_2\,\vec{\eta}_2 + l_3\vec{\eta}_3, \\
l_i &= 1,...L_i, \quad p = 0,1,
\end{aligned}\tag{A13}$$

where

$$\begin{aligned}
\vec{\eta}_p &= \tfrac{1}{\sqrt{3}}\left(1,1,1\right), \\
\vec{\eta}_1 &= \tfrac{2}{\sqrt{3}}\left(0,1,1\right), \\
\vec{\eta}_2 &= \tfrac{2}{\sqrt{3}}\left(1,0,1\right), \\
\vec{\eta}_3 &= \tfrac{2}{\sqrt{3}}\left(1,1,0\right).
\end{aligned}\tag{A14}$$

The total number of sites on the diamond lattice is $N_s = 2L_1L_2L_3$, and the volume per site is $v_s = \frac{8}{3\sqrt{3}}$ (in unit of a^3). The variable p can be interpreted as the parity of the corresponding lattice site: sites with the same parity are connected by an even number of links.

The two sublattices identified by $\vec{x}_+(l_1,l_2,l_3) \equiv \vec{x}(l_1,l_2,l_3,0)$ and $\vec{x}_-(l_1,l_2,l_3) \equiv \vec{x}(l_1,l_2,l_3,1)$ form two f.c.c. lattices having $N_s' = N_s/2$ sites. Each link of the diamond lattice connects sites belonging to different sublattices. It is convenient to rewrite a field $\phi(\vec{x}) \equiv \phi(l_1,l_2,l_3,p)$ in terms of two new fields $\phi_+(\vec{x}_+) \equiv \phi(\vec{x}_+)$ and $\phi_-(\vec{x}_-) \equiv \phi(\vec{x}_-)$ defined respectively on the sublattices $\vec{x}_+$ and $\vec{x}_-$. A finite lattice Fourier transform can be consistently defined by

$$\begin{aligned}
\phi_\pm(\vec{k}) &= \sum_{\vec{x}_\pm} e^{i\vec{k}\cdot\vec{x}_\pm}\,\phi_\pm(\vec{x}_\pm), \\
\phi_\pm(\vec{x}_\pm) &= \frac{1}{N_s'}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}_\pm}\phi_\pm(\vec{k}),
\end{aligned}\tag{A15}$$

and the set of momenta is

$$\vec{k} = \frac{2\pi}{L_1} m_1 \vec{\rho}_1 + \frac{2\pi}{L_2} m_2 \vec{\rho}_2 + \frac{2\pi}{L_3} m_3 \vec{\rho}_3,$$
$$m_i = 1, ... L_i, \tag{A16}$$

where

$$\begin{aligned}
\vec{\rho}_1 &= \tfrac{\sqrt{3}}{4}(-1, 1, 1), \\
\vec{\rho}_2 &= \tfrac{\sqrt{3}}{4}(1, -1, 1), \\
\vec{\rho}_3 &= \tfrac{\sqrt{3}}{4}(1, 1, -1),
\end{aligned} \tag{A17}$$

so that

$$\vec{k} \cdot \vec{x}' = \frac{2\pi}{L_1} m_1 l_1 + \frac{2\pi}{L_2} m_2 l_2 + \frac{2\pi}{L_3} m_3 l_3. \tag{A18}$$

The large-N limit of the two-point function is the massive Gaussian propagator defined on the same lattice. By using the Fourier transform (A16), straightforward calculations allow to derive a rather simple expression of the massive Gaussian propagator, and therefore of the large-N two-point function:

$$\begin{aligned}
G(\vec{x}; \vec{y}) &= G(\vec{x}', p_x; \vec{y}', p_y) = \\
&= \frac{3}{4\beta} \frac{1}{N_s'} \sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}'-\vec{y}')} \frac{1}{\Delta(k) + z\left(1 + \frac{1}{12}z\right)} \begin{pmatrix} 1 + \frac{1}{6}z & e^{-ik_1} H(k)^* \\ e^{ik_1} H(k) & 1 + \frac{1}{6}z \end{pmatrix}
\end{aligned} \tag{A19}$$

where

$$H(k) = \prod_{i=1}^{3} \cos \frac{k_i}{\sqrt{3}} - i \prod_{i=1}^{3} \sin \frac{k_i}{\sqrt{3}}, \tag{A20}$$

and

$$\begin{aligned}
\Delta(k) &= 3\left(1 - |H(k)|^2\right) = \\
&= \frac{3}{4}\left(3 - \cos \tfrac{2k_1}{\sqrt{3}} \cos \tfrac{2k_2}{\sqrt{3}} - \cos \tfrac{2k_2}{\sqrt{3}} \cos \tfrac{2k_3}{\sqrt{3}} - \cos \tfrac{2k_3}{\sqrt{3}} \cos \tfrac{2k_1}{\sqrt{3}}\right).
\end{aligned} \tag{A21}$$

Notice that $\Delta(k)$ has the same structure of the inverse propagator of the f.c.c. lattice, cfr. Eq. (A8). One can easily verify that $z = M_G^2$, where M_G is the second-moment mass. Using Eq. (A19) one can derive the expression of the lowest moments of $G(x)$ reported in Table VI.

The relation between β and z is determined by the condition $G(0) = 1$. In the infinite volume limit one can write

$$\beta = \frac{3}{4} \int_{-\pi}^{\pi} \frac{d^3 q}{(2\pi)^3} \frac{1 + \frac{1}{6}z}{\frac{3}{4}D(q) + z\left(1 + \frac{1}{12}z\right)}, \tag{A22}$$

where $D(q)$ has been already defined in Eq. (A11). Comparing Eqs. (A22) and (A10) for $z = 0$, one notes that $\beta_c^{\text{diamond}} = 4\beta_c^{\text{f.c.c.}}$. Therefore for the diamond lattice $\beta_c = 0.448220....$

In order to define a mass-gap estimator on the diamond lattice, one may consider the wall-wall correlation function defined constructing walls orthogonal to $\vec{w} = \frac{1}{\sqrt{2}}(-1, 1, 0)$, which is the direction orthogonal to two among the links starting from a site. We define

$$G_w(t \equiv \vec{x} \cdot \vec{w}) = \sum_{t=\text{cst}} G(\vec{x}) \tag{A23}$$

where the sum is performed over all sites with the same $t \equiv \vec{x} \cdot \vec{w} = \frac{2}{\sqrt{3}}(l_1 - l_2)$ (the coordinates of the sites $\vec{x}$ are given in Eq. (A13)). Using Eq. (A19) one can easily prove that $G_w(t)$ enjoys the property of exponentiation. The mass-gap μ can be extracted from the long-distance behavior of $G_w(t)$. For $t \gg 1$

$$G_w(t) \propto e^{-\mu t} \tag{A24}$$

In view of a strong-coupling analysis, it is convenient to use the following quantity

$$M_d^2 \equiv \frac{4}{3} \left(\cosh\sqrt{\tfrac{3}{2}}\mu - 1 \right) \tag{A25}$$

which has the property $M_d \to \mu$ for $\mu \to 0$ and has a regular strong-coupling series. In the large-N limit and for $\beta \leq \beta_c$

$$\frac{M_d^2}{M_G^2} = 1. \tag{A26}$$

APPENDIX B: PERTURBATIVE EXPANSION OF SCALING FUNCTIONS

In this appendix we present a simple derivation of all the results that are needed in order to construct explicitly the $1/N$, and g- and ϵ-expansions up to three loops presented in Sec. III. Our starting point is the observation that most of the two- and three-loop calculations needed in the relevant perturbative calculations are included, apart from rather trivial algebraic dependences on N, in the one-loop calculation of the $1/N$ expansion for the two-point function. As we shall show, the $1/N$ results can be expanded in g and ϵ in order to recover all the corresponding contributions. Let's therefore start with the evaluation of the renormalized self-energy to $O(1/N)$ in arbitrary dimension d and for arbitrary bare coupling g_0 in the N-component ϕ^4 theory.

We introduce the dressed composite propagator (geometric sum of bubble insertions in the ϕ^4 vertex):

$$\Delta^{-1}(y, g_0) \equiv \left[\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \frac{1}{(p+k)^2 + m^2} + \frac{3}{Ng} \right] m^{4-d} \equiv \Delta_r^{-1}(y) + \frac{3}{Ng}, \tag{B1}$$

where $y \equiv k^2/m^2$, and we have defined the (zero-momentum subtracted) dimensionless renormalized dressed (inverse) propagator:

$$\Delta_r^{-1}(y) \equiv \Delta^{-1}(y, g_0) - \Delta^{-1}(0, g_0)$$
$$= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \left[\frac{1}{(p+k)^2 + m^2} - \frac{1}{p^2 + m^2} \right], \tag{B2}$$

and the four-point (large-N) coupling renormalized at zero momentum

$$\frac{3}{Ng} = \Delta^{-1}(0, g_0) = \frac{\Gamma(2 - \frac{d}{2})}{2(4\pi)^{d/2}} + \frac{3m^{4-d}}{Ng_0} \equiv \frac{\Gamma\left(2 - \frac{d}{2}\right)}{2(4\pi)^{d/2}} \frac{N + 8}{N\bar{g}}, \tag{B3}$$

where we have rescaled the coupling for convenience, generalizing a rather standard three-dimensional prescription. The integration (B2) can be explicitly performed, and one obtains

$$\Delta_r^{-1}(y) = \frac{1}{2} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \left[\left(1 + \frac{y}{4}\right)^{d/2-2} F\left(2 - \frac{d}{2}, \frac{1}{2}, \frac{3}{2}, \frac{y}{y+4}\right) - 1 \right] \equiv \frac{\Gamma\left(2 - \frac{d}{2}\right)}{2(4\pi)^{d/2}} \delta_r(y), \quad \text{(B4)}$$

which is a regular function of d for all $d \leq 4$.

The renormalized $1/N$ contribution to the self-energy can now be evaluated by the formal expression

$$\phi_1(y, g) = \sigma(y, g) - \sigma(0, g) - y\frac{\partial}{\partial y}\sigma(y, g)|_{y=0}, \tag{B5}$$

$$\sigma(y, g) = m^{2-d} \frac{2(4\pi)^{d/2}}{\Gamma\left(2 - \frac{d}{2}\right)} \int \frac{d^d p}{(2\pi)^d} \frac{\bar{g}}{1 + \bar{g}\delta_r(p^2/m^2)} \frac{1}{(p+k)^2 + m^2}, \tag{B6}$$

and the subtractions that are symbolically indicated in Eq. (B5) must be done before performing the integration in Eq. (B6) in order to obtain finite quantities in all steps of the derivation. To this purpose, it is convenient to perform first the angular integration, by noticing that

$$\frac{2(4\pi)^{d/2}}{\Gamma\left(2 - \frac{d}{2}\right)} \int \frac{d^d p}{(2\pi)^d} \frac{m^{2-d}}{(p+k)^2 + m^2} f(p^2/m^2) = 2B\left(\tfrac{d}{2}, 2 - \tfrac{d}{2}\right)^{-1} \int_0^\infty (z)^{d/2-1} \, dz f(z) h(z, y)$$

$$\tag{B7}$$

where

$$h(z, y) = \frac{2}{B\left(\frac{d-1}{2}, \frac{1}{2}\right)} \int_0^\pi d\theta \frac{(\sin \theta)^{d-1}}{z + y + 1 + 2\sqrt{zy}\cos\theta}. \tag{B8}$$

The subtraction indicated in Eq. (B5) now simply amounts to replacing in Eq. (B6)

$$h(z, y) \longrightarrow h(z, y) - h(z, 0) - y\frac{\partial}{\partial y}h(z, y)|_{y=0}$$

$$= h(z, y) - \frac{1}{1 + z} + \frac{y}{(1 + z)^2} - \frac{4zy}{d(1 + z)^3}. \tag{B9}$$

The relevant results for some integer values of d are:

$$\delta_r(y) = \frac{4}{(4 + y)} - 1,$$

$$h(z, y) = \frac{z + y + 1}{(z + y + 1)^2 - 4zy} \tag{B10}$$

for $d = 1$;

$$\delta_r(y) = \frac{2}{y\xi} \ln \frac{\xi+1}{\xi-1} - 1,$$

$$h(z, y) = \left[(z + y + 1)^2 - 4zy\right]^{-1/2} \tag{B11}$$

for $d = 2$; here $\xi = \sqrt{1 + \frac{4}{y}}$;

$$\delta_r(y) = \frac{2}{\sqrt{y}} \arctan \frac{\sqrt{y}}{2} - 1,$$

$$h(z, y) = \frac{1}{4\sqrt{zy}} \ln \frac{z + y + 1 + 2\sqrt{zy}}{z + y + 1 - 2\sqrt{zy}} \tag{B12}$$

for $d = 3$;

$$\lim_{d \to 4} \Gamma\left(2 - \tfrac{d}{2}\right) \delta_r(y) = -\left(\xi \ln \frac{\xi+1}{\xi-1} - 2\right),$$

$$\lim_{d \to 4} h(z, y) = \frac{1}{2zy} \left[z + y + 1 - \sqrt{(z + y + 1)^2 - 4zy}\right]. \tag{B13}$$

Eqs. (B5) and (B6) are now ready for our purposes. Let us immediately notice that, since we know the leading large-N fixed point value of g, which happens to coincide with the $Ng_0 \to \infty$ limit of Eq. (B3), we may replace

$$\bar{g} \longrightarrow \bar{g}^* = 1, \tag{B14}$$

and find the $1/N$ contribution to the scaling function $\hat{g}_0(y)$, which in turn is simply the continuum non-linear σ model evaluation of the self-energy. This is the way Eq. (68) is generated, by setting $d = 3$ in the general expression. Eq. (B13) shows that, at least in the naive $d \to 4$ limit, there is no $O(1/N)$-non-Gaussian contribution to the self-energy scaling function.

Eq. (B6) is also the starting point for the g- and ϵ-expansion up to three loops. It is indeed straightforward to obtain a representation of the leading $O(1/N)$ contributions to the self-energy as a power series in g:

$$\phi_1(y, \bar{g}) = -\bar{g}^2 \widetilde{\varphi}_2(y) + \bar{g}^3 \widetilde{\varphi}_3(y) + O(\bar{g}^4), \tag{B15}$$

where we have defined the functions

$$\widetilde{\varphi}_n(y) = (-1)^n \frac{2}{B\left(\frac{d}{2}, 2 - \frac{d}{2}\right)} \int_0^\infty z^{\frac{d}{2}-1} dz [\delta_r(z)]^{n-1} \left[h(z, y) - \frac{1}{1+z} + \frac{y}{(1+z)^2} - \frac{4zy}{d(1+z)^3}\right], \tag{B16}$$

and we exploited the trivial consequence of the definition Eq. (B16): $\widetilde{\varphi}_1(y) \equiv 0$. Restoring the correct dependence on N for arbitrary (and not only very large) values of N in front of

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the functions φ_2 and φ_3 is now simply a combinatorial problem, whose solution leads to the complete three-loops result

$$f(y,\bar{g}) = 1 + y + \bar{g}^2 \frac{N+2}{(N+8)^2} \tilde{\varphi}_2(y) + \bar{g}^3 \frac{(N+2)}{(N+8)^2} \tilde{\varphi}_3(y) + O(\bar{g}^4) \tag{B17}$$

We must keep in mind that the functions $\tilde{\varphi}_n(y)$ carry a dependence on the dimensionality d, and the scaling function $\hat{g}_0(y)$ is the value taken by $f(y,\bar{g})$ when evaluated at the fixed point value $\bar{g}^*$ of the renormalized coupling, where $\bar{g}^*$ is in turn a function of the dimensionality and it is obtained by evaluating the zero of the β-function. We may now choose two different strategies. The first simply amounts to fixing d to the physical value we are interested in and replacing $\bar{g}^*$ with the numerical value (possibly evaluated by a higher-order expansion of the β-function at fixed dimension). We may however decide to expand the functions $\tilde{\varphi}_n(y)$ in the parameter $\epsilon \equiv 4 - d$ around their value at $d = 4$, perform a similar expansion for the β-function, then finding g^* as a series in ϵ [65]:

$$g^* = \Gamma(d/2)(4\pi)^{d/2} \frac{3}{N+8} \epsilon \left[1 + \epsilon \left(\frac{1}{2} + \frac{3(3N+14)}{(N+8)^2} \right) \right] + O(\epsilon^3). \tag{B18}$$

The functions $\varphi_2(y)$ and $\varphi_3(y)$ we have introduced in Sec. III C are strictly related to $\tilde{\varphi}_2(y)$ and $\tilde{\varphi}_3(y)$ calculated for $d = 3$, indeed

$$\begin{aligned} \varphi_2(y) &= \tilde{\varphi}_2(y)|_{d=3} \\ \varphi_3(y) &= [\tilde{\varphi}_3(y) - 2\tilde{\varphi}_2(y)]_{d=3} \,. \end{aligned} \tag{B19}$$

APPENDIX C: $O(1/N)$ CALCULATION OF $\hat{g}_4(y)$

In this appendix we present the calculation to order $1/N$ of the scaling function $\hat{g}_4(y)$. We will first study the small-p behaviour of the auxiliary propagator

$$\Delta^{-1}(p) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{[\overline{q+p}^2 + M_G^2][\overline{q}^2 + M_G^2]}. \tag{C1}$$

We are interested in the behaviour for $p \to 0$, $M_G \to 0$ with arbitrary ratio p/M_G. For $d < 4$, the only case we will consider, the leading order is simply given by the continuum expression

$$\Delta_0^{-1}(p) = \frac{1}{2} \int_{cont} \frac{d^d q}{(2\pi)^d} \frac{1}{[(q+p)^2 + M_G^2] (q^2 + M_G^2)}. \tag{C2}$$

Here and in the following, when we will append the subscript "cont" to the integral, we will mean that the integration domain is the whole d-dimensional space; otherwise the integration should be intended over the first Brillouin zone. We want now to compute the first correction. Using

$$\overline{q}^2 = q^2 + e_{0,2}(q^2)^2 + e_{4,0}Q_4(q) + O(q^6), \tag{C3}$$

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we could expand (C1) in powers of q. This however cannot be the correct answer as the integral one obtains in this way is ultraviolet-divergent. There is however a standard way out. Introduce a fixed cut-off Λ and define a sharp-momentum regularized quantity

$$I(q, p; M_G, \Lambda) = \frac{1}{((q+p)^2 + M_G^2)(q^2 + M_G^2)^2} \times$$
$$\left[q^2 + M_G^2 - 2(e_{0,2}(q^2)^2 + e_{4,0}Q_4(q)) + 2e_{0,2}(q^2 + M_G^2)^2\theta((q+p)^2 - \Lambda^2)\right] \tag{C4}$$

where $\theta(x)$ is the step-function $[\theta(x) = 1$ for $x \geq 0$, $\theta(x) = 0$ for $x < 0]$. Then we rewrite (C1) as

$$\Delta^{-1}(p) = \frac{1}{2}\int_{cont}\frac{d^dq}{(2\pi)^d}\left[\frac{\chi_{latt}(q)}{[\overline{q+p}^2 + M_G^2][\overline{q}^2 + M_G^2]} - I(q, p; M_G, \Lambda)\right]$$
$$+ \frac{1}{2}\int_{cont}\frac{d^dq}{(2\pi)^d}I(q, p; M_G, \Lambda) \tag{C5}$$

where $\chi_{latt}(q)$ is the characteristic function of the first Brillouin zone, i.e. $\chi_{latt}(q) = 1$ if $q \in [-\pi, \pi]^d$, $\chi_{latt}(q) = 0$ otherwise. For $p \to 0$ and $M_G \to 0$ (but only for $2 \leq d < 4$), we can simply set $M_G = p = 0$ in the first integral thus obtaining a constant which depends on Λ and on the specific lattice Hamiltonian. We must now compute the second integral. A completely straightforward calculation gives

$$\frac{1}{2}\int_{cont}\frac{d^dq}{(2\pi)^d}I(q, p; M_G, \Lambda) = \Delta_0^{-1}(p) - e_{0,2}\left[\int_{q^2 \leq \Lambda^2}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2 + M_G^2} - 4M_G^2\Delta_0^{-1}(p) - M_G^4\frac{\partial\Delta_0^{-1}(p)}{\partial p^2}\right]$$
$$- e_{4,0}\frac{Q_4(p)}{(p^2)^2}\left[A(p^2) + B(p^2)\Delta_0^{-1}(p)\right] \tag{C6}$$

where

$$A(p^2) = \frac{(4\pi)^{-d/2}\Gamma(3 - d/2)}{2(d-1)(d-4)}\frac{p^2(d+8) + 36M_G^2}{p^2 + 4M_G^2}M_G^{d-2}, \tag{C7}$$

$$B(p^2) = \frac{1}{8(d-1)}\frac{1}{p^2 + 4M_G^2}\left[(4-d)(d+2)(p^2)^2 - 8(d-10)p^2M_G^2 + 144M_G^4\right]. \tag{C8}$$

Summing the two terms we can rewrite the auxiliary propagator as

$$\Delta^{-1}(p) \approx \Delta_0^{-1}(p) + C(M_G) + e_{0,2}f_0(p^2, M_G) + e_{4,0}Q_4(p)f_4(p^2, M_G), \tag{C9}$$

where

$$C(M_G) = \frac{1}{2}\int_{cont}\frac{d^dq}{(2\pi)^d}\left[\frac{\chi_{latt}(q)}{(\overline{q}^2)^2} - \frac{1}{(q^2)^2} + 2e_{0,2}\left(\frac{1}{q^2} - \frac{1}{q^2 + M_G^2}\right)\right], \tag{C10}$$

$$f_0(p^2, M_G) = -M_G^2\left(4\Delta_0^{-1}(p) + M_G^2\frac{\partial\Delta_0^{-1}(p)}{\partial p^2}\right), \tag{C11}$$

$$f_4(p^2, M_G) = -\frac{1}{(p^2)^2}\left(A(p^2) + B(p^2)\Delta_0^{-1}(p)\right). \tag{C12}$$

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The functions $f_0(p^2, M_G)$ and $f_4(p^2, M_G)$ are universal: they do not depend on the lattice structure. The lattice dependence is contained in the two constants $e_{0,2}$ and $e_{4,0}$ and in $C(M_G)$ which depends on the explicit form of the lattice hamiltonian.

Let us now consider $\tilde{G}^{-1}(k, M_G)$ to order $1/N$. We will first compute $Z_4(M_G) \equiv g_4(0, M_G)$ which can be easily obtained from

$$Z_4(M_G) = \frac{d+2}{24d(d-1)} Q \left(\partial/\partial k\right) \tilde{G}^{-1}(k)|_{k=0}. \tag{C13}$$

Using the expression for $\tilde{G}^{-1}(k)$ to order $1/N$, cfr. formula (137), we get

$$Z_4(M_G) = e_{4,0} + \frac{1}{N} \frac{d+2}{24d(d-1)} \int \frac{d^d q}{(2\pi)^d} \Delta(q) \delta z_4(q) \tag{C14}$$

where

$$\begin{aligned}
\delta z_4(q) = & -\frac{1}{D(q)^2} \left[\frac{d-1}{d+2} \sum_\mu w_{\mu\mu\mu\mu} - \frac{3}{d+2} \sum_{\mu\neq\nu} w_{\mu\mu\nu\nu} \right] \\
& + \frac{8}{D(q)^3} \left[\frac{d-1}{d+2} \sum_\mu w_{\mu\mu\mu} w_\mu - \frac{3}{d+2} \sum_{\mu\neq\nu} w_{\mu\mu\nu} w_\nu \right] \\
& - \frac{36}{D(q)^4} \left[\frac{d-1}{d+2} \sum_\mu w_{\mu\mu} w_\mu^2 - \frac{1}{d+2} \sum_{\mu\neq\nu} w_{\mu\mu} w_\nu^2 \right] \\
& + \frac{6}{D(q)^3} \left[\frac{d-1}{d+2} \sum_\mu w_{\mu\mu}^2 - \frac{1}{d+2} \sum_{\mu\neq\nu} (w_{\mu\mu} w_{\nu\nu} + 2w_{\mu\nu}^2) \right] \\
& + \frac{72}{D(q)^4} \frac{1}{d+2} \sum_{\mu\neq\nu} w_{\mu\nu} w_\mu w_\nu \\
& + \frac{24}{D(q)^5} \left[\frac{d-1}{d+2} \sum_\mu w_\mu^4 - \frac{3}{d+2} \sum_{\mu\neq\nu} w_\mu^2 w_\nu^2 \right] \quad . \tag{C15}
\end{aligned}$$

Here $D(q) = \bar{q}^2 + M_G^2$, $w_\mu = \partial_\mu \bar{q}^2$, $w_{\mu\nu} = \partial_\nu \partial_\mu \bar{q}^2$ and so on. From this expression one can easily compute the exponent η_4: indeed one must expand $\delta z_4(q)$ and $\Delta(q)$ for $q \to 0$ and keep only those terms that behave as $\Delta_0(q)(q^2 + M_G^2)^{-2}$. We obtain in this way the results (57-122) for $l = 2$.

We want now to compute the scaling function $\hat{g}_4(y)$. First of all notice that

$$g_4(y, M_G) = \frac{1}{\mathcal{N}_4} \int d^d \Omega(\hat{k}) \frac{Q_4(k)}{(k^2)^4} \tilde{G}^{-1}(k, M_G) \tag{C16}$$

where $d^d \Omega(\hat{k})$ indicates the normalized measure on the $(d-1)$-dimensional sphere and we obtained

$$\int d^d \Omega(\hat{k}) Q_4(k)^2 = \frac{24(d-1)}{(d+2)^2(d+4)(d+6)} (k^2)^4 \equiv \mathcal{N}_4 (k^2)^4, \tag{C17}$$

$$\int d^d \Omega(\hat{k}) Q_6(k)^2 = \frac{720(d-2)(d-1)}{(d+2)(d+4)^2(d+6)(d+8)^2(d+10)} (k^2)^6 \equiv \mathcal{N}_6 (k^2)^6. \tag{C18}$$

(We shall not need Eq. (C18), but we quote it for further reference.) Using then Eq. (137) we get

$$g_4(y, M_G) = e_{4,0} + \frac{1}{N}\frac{1}{\mathcal{N}_4} \int d^d\Omega(\hat{k})\frac{Q_4(k)}{(k^2)^4} \int \frac{d^dq}{(2\pi)^d}\frac{\Delta(q)}{(q+k)^2 + M_G^2}, \tag{C19}$$

so that

$$\hat{g}_4(y) = 1 + \frac{1}{N}\frac{1}{e_{4,0}\mathcal{N}_4}\lim_{M_G^2 \to 0} \int d^d\Omega(\hat{k})\frac{Q_4(k)}{(k^2)^4} \times$$

$$\left\{\int \frac{d^dq}{(2\pi)^d}\left[\frac{\Delta(q)}{(q+k)^2 + M_G^2} - \frac{d+2}{24d(d-1)}Q_4(k)\Delta(q)\delta z_4(q)\right]\right\}. \tag{C20}$$

Now because of the subtraction, we can take the limit $M_G \to 0$ by expanding the integrand in powers of q and using (C9). We now integrate over the angular variables. In particular we need to compute the following two integrals:

$$I_1(k^2, q^2, M_G^2) = \frac{1}{\mathcal{N}_4}\int d^d\Omega(\hat{k})\int d^d\Omega(\hat{q})\frac{Q_4(k)Q_4(q)}{(q+k)^2 + M_G^2}, \tag{C21}$$

$$I_2(k^2, q^2, M_G^2) = \frac{1}{\mathcal{N}_4}\int d^d\Omega(\hat{k})\int d^d\Omega(\hat{q})\frac{Q_4(k)Q_4(q+k)}{((q+k)^2 + M_G^2)^2}. \tag{C22}$$

Using the techniques of Ref. [10] we get

$$I_1(k^2, q^2, M_G^2) = (k^2q^2)^{3/2}F_{d,4}(z), \tag{C23}$$

$$I_2(k^2, q^2, M_G^2) = -\frac{(k^2)^3}{2q^2}\left[F'_{d,0}(z) + 4\left(\frac{q^2}{k^2}\right)^{1/2}F'_{d,1}(z) + \frac{6q^2}{k^2}F'_{d,2}(z)+\right.$$

$$\left. 4\left(\frac{q^2}{k^2}\right)^{3/2}F'_{d,3}(z) + \left(\frac{q^2}{k^2}\right)^2 F'_{d,4}(z)\right], \tag{C24}$$

where we have defined:

$$z = \frac{q^2 + k^2 + M_G^2}{2\sqrt{q^2k^2}}, \tag{C25}$$

$$F_{d,l}(z) = \frac{2^{(1-d)/2}l!(d-2)!}{\Gamma((d-1)/2)(d+l-3)!}(-1)^l e^{-(d-3)\pi i/2}(z^2 - 1)^{(d-3)/4}Q^{(d-3)/2}_{l+(d-3)/2}(z). \tag{C26}$$

Here $Q^\nu_\mu(z)$ is the associated Legendre function of the second kind (see Ref. [66], Sec. 8.7 and 8.8). Notice that for $l = 0$ the functions $F_{d,0}(z)$ are related to $h(z,y)$ defined in (B8). Indeed

$$F_{d,0}(z) = \sqrt{q^2k^2}\, h\left(\frac{q^2}{M_G^2}, \frac{k^2}{M_G^2}\right). \tag{C27}$$

In two dimensions it is easy to see that

$$F_{2,l}(z) = \frac{1}{2\sqrt{z^2-1}}\left(\sqrt{z^2-1}-z\right)^l. \tag{C28}$$

For $d=3$ we have

$$F_{3,l}(z) = \frac{(-1)^l}{4}Q_l(z), \tag{C29}$$

and for $d=4$

$$F_{4,l}(z) = -\frac{1}{l+1}\left(\sqrt{z^2-1}-z\right)^{l+1}. \tag{C30}$$

Putting all together, we get finally

$$\widehat{g}_4(y) = 1 - \frac{1}{N}\frac{1}{y^4}\int\frac{d^dq}{(2\pi)^d}\left[\Delta_0(q)^2 f_4(q^2,1)I_1(y,q^2,1) + \Delta_0(q)I_2(y,q^2,1) - \text{subtr}\right] \tag{C31}$$

where "subtr" indicates the integrand computed for $y\to 0$. As expected the final result is universal: any dependence on the lattice hamiltonian has disappeared.

APPENDIX D: STRONG-COUPLING EXPANSION OF $G(x)$ ON THE CUBIC LATTICE

Presenting l-th order strong-coupling results for the two-point Green's function would naively imply writing down as many coefficients as the number of lattice sites that can be reached by a l-step random walk starting from the origin (up to discrete lattice symmetries). It is interesting to notice the relationship existing between the number n_l of lattice points (not related by a lattice symmetry) that lie at a given lattice distance l from the origin and the number of independent lattice-symmetric functions $Q_{2m}(k)(k^2)^{l-m}$. One can easily get convinced that, on a hypercubic lattice, the number of functions $Q_{2m}^{(p)}(k)(k^2)^{l-m}$ is the same as the number of monomials of total degree l in the d variables k_i^2 that are not related by a lattice symmetry (that is, the number of independent, homogeneous lattice-symmetric degree-l polynomials in the k_i^2). This number in turn is equal to that of the partitions of l into d ordered non-negative integers, and this is nothing but the number of independent lattice points at a lattice distance l (where ordering insures independence by elimination of copies obtained by permutation). As a corollary, the relationship $p_l = n_l - n_{l-1}$ holds for arbitrary d on hypercubic lattices. A generating function for n_l, for a given value of d, is

$$\sum_{l=0}^{\infty}n_l t^l = \prod_{n=1}^{d}\frac{1}{1-t^m}, \tag{D1}$$

implying the asymptotic behavior

$$n_l \longrightarrow \frac{l^{d-1}}{d!(d-1)!}. \tag{D2}$$

In the case of three-dimensional hypercubic lattices, one can show that $p_l = \lfloor l/6\rfloor + 1$ with the exception of $l = 6k+1$ in which case $p_l = k$, while n_l is the integer nearest to

$(l + 3)^2/12$ and the sum $\sum_{i=\text{even}}^{l} n_i$ is the integer nearest to $(l + 4)^3/72$. This would mean roughly $(l + 4)^3/72$ coefficients for the l-th order of the strong-coupling expansion on the cubic lattice. This number can be sensibly reduced (asymptotically by a factor 27 on the cubic lattice), without losing any physical information, by noticing that the inverse two-point function, when represented in coordinate space, involves only points that can be reached by a $\lfloor l/3 \rfloor$-step random walk. This fact can be traced to the one-particle irreducible nature of the inverse correlation. As a matter of fact, instead of the 93 coefficients needed to represent the 15-th order contributions to $G(x)$, only 8 coefficients are enough for the corresponding contribution to the inverse function $G^{-1}(x)$, which we construct by the following procedure (a similar representation was used for the Ising model in a magnetic field in [6]).

We introduce the equivalence classes of lattice sites under symmetry transformations and choose a representative y for each class: whenever $x \sim y$ then $G(x) = G(y)$. We define the "form factor" of the equivalence class

$$Q(y) = \sum_{x \sim y} e^{ipx}, \tag{D3}$$

and represent the Fourier transform of $G(x)$ according to

$$\widetilde{G}(p) = \sum_{y} Q(y)G(y). \tag{D4}$$

The momentum-space inverse Green's function is defined by

$$\widetilde{G}^{-1}(p)\widetilde{G}(p) = 1. \tag{D5}$$

Therefore its inverse Fourier transform enjoys the symmetries of $G(x)$ and satisfies the relationships

$$\widetilde{G}^{-1}(p) = \sum_{x} e^{ipx} G^{-1}(x) = \sum_{y} Q(y)G^{-1}(y). \tag{D6}$$

In practice we exploit the property

$$Q(v)Q(y) = \sum_{z} n(z; v, y)Q(z) \tag{D7}$$

where

$$n(z; v, y) = \sum_{u \sim v, x \sim y} \delta_{z, u+x} \tag{D8}$$

are integer numbers, and reduce the problem of evaluating $G^{-1}(y)$ to that of solving the linear system of equations

$$\sum_{v} G^{-1}(v)M(v, z) = \delta_{z,0} \tag{D9}$$

where

$$M(v, z) = \sum_{y} G(y)n(z; v, y). \tag{D10}$$

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When expanding in powers of β, the system takes a triangular structure and , as expected, it admits a solution whose non-trivial terms are only those corresponding to the equivalence classes of sites that can be reached by $l/3$ random steps.

Solutions for $G^{-1}(x)$ can be found for arbitrary N. In Table XI we only exhibit $G^{-1}(x)$ for $N = 0, 1, 2, 3, 4$ and 16. We choose a representative of the equivalence class by the prescription $x_1 \geq x_2 \geq x_3 \geq 0$. We may notice as a general feature that in the class represented by $x_1 > 1$, $x_2 = x_3 = 0$ the first non-trivial contribution is of order $3x_1 + 2$ ($3x_1 + 4$ when $N = 1$). When $N = 0, 1$ a number of seemingly non-trivial coefficients turn out to be zero.

APPENDIX E: STRONG-COUPLING SERIES OF χ AND m_2 ON THE DIAMOND LATTICE

On the diamond lattice we have calculated the strong-coupling expansion of $G(x)$ up to 21st order. In the character-like approach, the possibility of reaching larger orders than on the cubic lattice is related to the smaller coordination number. However longer series do not necessarily mean that more precise results can be obtained from their analysis. This is essentially related to the approach to the asymptotic regime of the correspoding strong coupling expansion, which is expected to occur later on lattices with smaller coordination number. The 21th order series on the diamond lattice provide estimates of the exponents γ and ν which are, as we shall see for $N = 1, 2, 3$, substantially consistent with the results obtained by analyzing series on cubic-like lattices (see for example Ref. [40] where series to $O(\beta^{21})$ for the cubic and b.c.c. lattice have been presented and analyzed), but less precise.

In this appendix we report the 21st order strong-coupling series of χ and m_2 calculated on the diamond lattice, for $N = 1, 2, 3$. 27th order strong-coupling series for $N = 0$, i.e. for the self-avoiding walk model, can be found in Ref. [42].

1. $N = 1$

$$
\chi = 1 + 4\beta + 12\beta^2 + \frac{104\beta^3}{3} + 100\beta^4 + \frac{4328\beta^5}{15} + \frac{12128\beta^6}{15} + \frac{711328\beta^7}{315} + \frac{132452\beta^8}{21} +
$$
$$
\frac{49894088\beta^9}{2835} + \frac{230044448\beta^{10}}{4725} + \frac{20986492048\beta^{11}}{155925} + \frac{11593048528\beta^{12}}{31185} +
$$
$$
\frac{6239638466896\beta^{13}}{6081075} + \frac{40044715794736\beta^{14}}{14189175} + \frac{381115667726672\beta^{15}}{49116375} + \frac{907261838473556\beta^{16}}{42567525} +
$$
$$
\frac{635228192216156408\beta^{17}}{10854718875} + \frac{5223277546855685888\beta^{18}}{32564156625} + \frac{81581590401875 6584288\beta^{19}}{1856156927625} +
$$
$$
\frac{744572898253973823856\beta^{20}}{618718975875} + \frac{6420209970515817366 73936\beta^{21}}{194896477400625} + O(\beta^{22}). \tag{E1}
$$

$$
m_2 = 4\beta + 32\beta^2 + \frac{488\beta^3}{3} + \frac{2048\beta^4}{3} + \frac{38888\beta^5}{15} + \frac{417664\beta^6}{45} + \frac{10027936\beta^7}{315} + \frac{33306368\beta^8}{315} +
$$

$$\frac{971601608\,\beta^9}{2835} + \frac{15453950464\,\beta^{10}}{14175} + \frac{532482065296\,\beta^{11}}{155925} + \frac{4939730085376\,\beta^{12}}{467775} +$$

$$\frac{196443743845456\,\beta^{13}}{6081075} + \frac{4168605624019328\,\beta^{14}}{42567525} + \frac{188065240470724112\,\beta^{15}}{638512875} +$$

$$\frac{561744708980235008\,\beta^{16}}{638512875} + \frac{28352355075085440248\,\beta^{17}}{10854718875} +$$

$$\frac{57966531061027107328\,\beta^{18}}{7514805375} + \frac{42081281751167641189216\,\beta^{19}}{1856156927625} +$$

$$\frac{615759333006052918694656\,\beta^{20}}{9280784638125} + \frac{37696556941296724618984336\,\beta^{21}}{194896477400625} + O(\beta^{22}). \qquad \text{(E2)}$$

We have analyzed the series of χ by using the $[m/l/k]$ first order IA's with

$$m + l + k + 2 = 21,$$
$$\lfloor (n-2)/3 \rfloor - 2 \le m, l, k \le \lceil (n-2)/3 \rceil + 2. \qquad \text{(E3)}$$

We have obtained $\beta_c = 0.3697(1)$ and $\gamma = 1.238(14)$. An estimate of γ can be also obtained by applying CPRM to the series χ^2 and χ, as explained in Sec. IV F. By employing biased IA's (those indicated in (E3)), one finds $\gamma = 1.253(4)$. By applying CPRM to the series m_2 and χ, and using biased IA's, one finds $\nu = 0.645(4)$. These values of γ and ν are slithly larger than the available estimates obtained by other tecniques (field-theoretical approaches give $\gamma \simeq 1.240$ and $\nu \simeq 0.630$), or strong coupling expansion on other lattices, but we would not consider them inconsistent. One should not forget that the error displayed does not take into account possible systematic errors (due, for example, to confluent singularities), but just the spread of the results of the various IA's indicated in (E3).

2. $N = 2$

$$\chi = 1 + 4\,\beta + 12\,\beta^2 + 34\,\beta^3 + 96\,\beta^4 + \frac{814\,\beta^5}{3} + 743\,\beta^6 + \frac{24145\,\beta^7}{12} + \frac{10925\,\beta^8}{2} +$$

$$\frac{889703\,\beta^9}{60} + \frac{2387483\,\beta^{10}}{60} + \frac{22968773\,\beta^{11}}{216} + \frac{25617551\,\beta^{12}}{90} +$$

$$\frac{11516036093\,\beta^{13}}{15120} + \frac{40849680041\,\beta^{14}}{20160} + \frac{520550507027\,\beta^{15}}{96768} +$$

$$\frac{3457894675397\,\beta^{16}}{241920} + \frac{495995794312009\,\beta^{17}}{13063680} +$$

$$\frac{2188572410969059\,\beta^{18}}{21772800} + \frac{173608313274399461\,\beta^{19}}{653184000} +$$

$$\frac{76543471229019871\,\beta^{20}}{108864000} + \frac{5344313242348050991\,\beta^{21}}{2874009600} + O(\beta^{22}). \qquad \text{(E4)}$$

$$m_2 = 4\,\beta + 32\,\beta^2 + 162\,\beta^3 + 672\,\beta^4 + \frac{7534\,\beta^5}{3} + \frac{26488\,\beta^6}{3} + \frac{356305\,\beta^7}{12} + \frac{289444\,\beta^8}{3} +$$

$$\frac{18326503\,\beta^9}{60} + \frac{42659326\,\beta^{10}}{45} + \frac{3125910649\,\beta^{11}}{1080} + \frac{1176454982\,\beta^{12}}{135} +$$

$$\frac{78423473449\,\beta^{13}}{3024} + \frac{577822206313\,\beta^{14}}{7560} + \frac{108069034519903\,\beta^{15}}{483840} +$$

$$\frac{58770348791597\,\beta^{16}}{90720} + \frac{24384512261505001\,\beta^{17}}{13063680} +$$

$$\frac{43660988509648999\,\beta^{18}}{8164800} + \frac{9954936929950180901\,\beta^{19}}{653184000} +$$

$$\frac{1764942584095467281\,\beta^{20}}{40824000} + \frac{175488325636140 3082267\,\beta^{21}}{14370048000} + O(\beta^{22}). \qquad \text{(E5)}$$

By performing an IA analysis of the series of χ, one finds $\beta_c = 0.3845(2)$ and $\gamma = 1.33(2)$. By applying CPRM to the series χ^2 and χ, and employing biased IA's, one finds $\gamma = 1.34(1)$. By applying CPRM to the series m_2 and χ, and using biased IA's, one finds $\nu = 0.689(8)$. These results are substantially consistent with the available estimates of γ obtained on other lattices and by other approaches (see e.g. Refs. [40] and [2]).

3. $N = 3$

$$\chi = 1 + 4\,\beta + 12\,\beta^2 + \frac{168\,\beta^3}{5} + \frac{468\,\beta^4}{5} + \frac{9144\,\beta^5}{35} + \frac{123456\,\beta^6}{175} + \frac{65568\,\beta^7}{35} + \frac{873708\,\beta^8}{175} +$$

$$\frac{128270568\,\beta^9}{9625} + \frac{11818853472\,\beta^{10}}{336875} + \frac{2007117038928\,\beta^{11}}{21896875} + \frac{5262987995856\,\beta^{12}}{21896875} +$$

$$\frac{1973906542032\,\beta^{13}}{3128125} + \frac{25696714370736\,\beta^{14}}{15640625} + \frac{277395071474138256\,\beta^{15}}{65143203125} +$$

$$\frac{5048136975344060076\,\beta^{16}}{456002421875} + \frac{174731287641977188 3176\,\beta^{17}}{60648322109375} +$$

$$\frac{3552388335040525307 8656\,\beta^{18}}{476522530859375} + \frac{32070882115870547276 72352\,\beta^{19}}{16678288580078125} +$$

$$\frac{82941862934668439888 64336\,\beta^{20}}{16678288580078125} + \frac{285612671193686662 161552\,\beta^{21}}{221862716796875} + O(\beta^{22}). \qquad \text{(E6)}$$

$$m_2 = 4\,\beta + 32\,\beta^2 + \frac{808\,\beta^3}{5} + \frac{3328\,\beta^4}{5} + \frac{17240\,\beta^5}{7} + \frac{1498496\,\beta^6}{175} + \frac{4978592\,\beta^7}{175} + \frac{15959296\,\beta^8}{175} +$$

$$\frac{391158744\,\beta^9}{1375} + \frac{292871549952\,\beta^{10}}{336875} + \frac{8170771755824\,\beta^{11}}{3128125} + \frac{169326765636096\,\beta^{12}}{21896875} +$$

$$\frac{495146153921968\,\beta^{13}}{21896875} + \frac{7166586778308992\,\beta^{14}}{109484375} + \frac{174778808251494500 8\,\beta^{15}}{9306171875} +$$

$$\frac{243755148694999429888\,\beta^{16}}{456002421875} + \frac{9162712230876234 5759912\,\beta^{17}}{60648322109375} +$$

$$\frac{2022510678813614989 101568\,\beta^{18}}{476522530859375} + \frac{19778750840713858434 5236512\,\beta^{19}}{16678288580078125} +$$

$$\frac{219940729786777629556242688\,\beta^{20}}{667131543203125} + \frac{3499869172501434663 1615751056\,\beta^{21}}{383600637341796875} + O(\beta^{22}). \qquad \text{(E7)}$$

By performing an IA analysis of the series of χ, one finds $\beta_c = 0.3951(2)$ and $\gamma = 1.42(2)$. We mention that singularities approximately as far to the origin as β_c have been detected by our analysis, indeed we found two singularities at $\beta \simeq \pm i0.39$. By applying CPRM to the series χ^2 and χ, and employing biased IA's, we obtained $\gamma = 1.42(1)$. By applying CPRM to the series m_2 and χ, and using biased IA's, one finds $\nu = 0.726(4)$. These results are slightly larger (and less precise) than the values obtained on other lattices (see e.g. Ref. [40], or by other techniques (see e.g. Ref. [2]), but substantially consistent.

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TABLES

TABLE I. Numerical values of the coefficients $h_i^{(n,j)}$ defined in Eq. (90).

i	$h_i^{(2)}$	$h_i^{(3)}$	$h_i^{(4,1)}$	$h_i^{(4,2)}$	$h_i^{(4,3)}$
2	$-\frac{2}{405}$	0.00949125	0.000765804	-0.0134856	-0.0345992
3	$\frac{4}{15309}$	-0.000612784	-0.00000341189	0.00102554	0.00253490
4	$-\frac{1}{59049}$	0.0000450060	-0.00000233206	-0.0000841861	-0.00020327
5	$\frac{4}{3247695}$	-0.0000035762	0.00000035769	0.0000072651	0.00002390

TABLE II. For several values of N and for the cubic and diamond lattice, we report the values of β_c we used in our strong-coupling calculations. We also report the fixed-point value $\bar{g}^*$ of the rescaled zero-momentum four-point renormalized coupling, as obtained by field-theoretical methods.

N	cubic	diamond	$\bar{g}^*$
0	$\beta_c =0.213492(1)$ [53]	$\beta_c =0.34737(1)$ [42]	1.421(8) [2]
1	$\beta_c =0.2216544(3)$ [47]	$\beta_c =0.3697(1)$	1.416(5) [2]
2	$\beta_c =0.22710(1)$ [48]	$\beta_c =0.3845(2)$	1.406(4) [2]
3	$\beta_c =0.231012(12)$ [49]	$\beta_c =0.3951(2)$	1.391(4) [2]
4	$\beta_c =0.23398(2)$ [40]	$\beta_c =0.4027(2)$	1.369 [22]
8	$\beta_c =0.24084(3)$ [40]	$\beta_c =0.4200(2)$	1.303 [22]
16	$\beta_c =0.24587(6)$	$\beta_c =0.4327(2)$	1.207 [22]
32	$\beta_c =0.2491(1)$	$\beta_c =0.4401(1)$	1.122 [22]
∞	$\beta_c =0.252731...$	$\beta_c =0.448220...$	1

TABLE III. Estimates of c_2, c_3, and $S_M - 1$ for selected values of N from various analyses of the strong-coupling series on the cubic lattice (see the text). An asterisk indicates that most of the approximants are defective, or, in the cases where numbers are not shown, that all approximants are defective, so that no estimate can be extracted.

	N	PS	PA	DPA	IA
$10^4 c_2$	0	$-0.6(1.8)$	$*-2(2)$	$*$	$-1.0(3)$
	1	$-3.0(4)$	$-3.2(5)$	$-3.0(6)$	$-2.9(1)$
	2	$-3.9(3)$	$-4.1(7)$	$-4.0(5)$	$-3.9(1)$
	3	$-4.1(1)$	$-4.5(1.2)$	$-4(3)$	$*-3.7(4)$
	4	-4.06	$-4.4(6)$	$-4.3(2)$	$-4.09(3)$
	8	$-3.4(1)$	$-3.6(6)$	$-3.59(9)$	$-3.5(1)$
	16	-2.26	$-2.5(3)$	$-2.4(1)$	$-2.43(7)$
	32	-1.28	$-1.5(2)$	$-1.46(9)$	$-1.45(4)$
$10^4 c_3$	0	0.257	$0.12(2)$	$*$	$0.125(3)$
	1	0.170	$0.11(4)$	$0.107(3)$	$0.10(2)$
	2	0.205	$0.11(2)$	$0.11(1)$	$0.11(2)$
	3	0.212	$0.11(2)$	$0.12(1)$	$*$
	4	0.133	$0.12(1)$	$0.12(1)$	$*$
	8	-0.286	$0.07(4)$	$0.10(1)$	$*$
	16	-0.582	$0.04(3)$	$0.070(5)$	$*$
	32	-0.559	$0.02(2)$	$0.041(3)$	$*0.03(2)$
$10^4 (S_M - 1)$	0	-2.03	$0(2)$	$*$	$*$
	1	-1.76	$-2(1)$	$-3(1)$	$*$
	2	-2.72	$-4(1)$	$*-2(4)$	$*-3.3(6)$
	3	-3.34	$-4(1)$	$-4(3)$	$-3.9(3)$
	4	-3.65	$-4.4(9)$	$-4(3)$	$-3.9(4)$
	8	-3.63	$-3.9(5)$	$-4(2)$	$-3.9(5)$
	16	-2.72	$-2.8(3)$	$-3(1)$	$-2.8(2)$
	32	-1.71	$*-1.8(3)$	$-2.0(7)$	$*$

TABLE IV. Estimates of c_2, c_3 and $S_M - 1$ from various analyses of the strong-coupling series on the diamond lattice. An asterisk indicates that most of the approximants considered are defective, or, in the cases where numbers are not shown, that all approximants are defective, so that no estimate can be extracted.

	N	PA	DPA	IA
$10^4 c_2$	0	$-2(1)$	*	$-0.6(3)$
	1	$-3.0(3)$	$-2.9(5)$	$-3.0(2)$
	2	$-4.1(4)$	$-4.1(5)$	$-4.36(4)$
	3	$-4.6(3)$	$-4.4(3)$	$*-4.7(1)$
	4	$-4.7(2)$	$*-4.7(3)$	$-4.8(2)$
	8	$-4.0(2)$	$*-3.9(2)$	$-3.97(3)$
	16	$-2.66(7)$	$*-2.6(1)$	$-2.65(2)$
	32	$-1.52(3)$	$*-1.51(7)$	$-1.49(5)$
$10^4 c_3$	0	$0.10(3)$	*	$0.099(6)$
	1	$0.12(2)$	$0.08(2)$	$0.11(2)$
	2	$0.12(4)$	$0.08(1)$	$0.12(2)$
	3	$0.12(8)$	$0.09(1)$	$0.13(2)$
	4	$0.1(2)$	$0.09(1)$	$0.12(2)$
	8	$0.2(4)$	$*0.08(2)$	$0.1(1)$
	16	$0.1(2)$	$0.06(2)$	$0.08(7)$
	32	$0.0(1)$	$0.04(1)$	*
$10^4 (S_M - 1)$	0	$0(1)$	*	*
	1	$-2.3(4)$	$-2.2(3)$	$-2.3(4)$
	2	$-3.6(4)$	$-3.4(2)$	$-3.5(2)$
	3	$-4.0(5)$	$-3.9(3)$	$*-4(1)$
	4	$-4.3(7)$	$-4.1(2)$	$*-5(3)$
	8	$-4.1(7)$	$-3.6(3)$	$-4.0(3)$
	16	$-3.0(4)$	$-2.4(2)$	$-2.8(2)$
	32	$-1.9(3)$	$-1.5(2)$	$-1.7(2)$

TABLE V. Estimates of the coefficients c_2 and c_3, and of the mass-ratio S_M, from the analysis of the strong-coupling series of $\bar{u}_2$ and $\bar{u}_3$, M_c^2/M_G^2 (on the cubic lattice) and M_d^2/M_G^2 (on the diamond lattice). We report also results from the $1/N$-expansion, from the g-expansion and the ϵ-expansion. In the latter cases we give two numbers corresponding to the two choices: resumming $R(x)$ or $R(x)/x^2$. For $N = 1$ we also give the estimates from an "improved" resummation of the ϵ-expansion which takes into account the exactly known results in two dimensions.

N		$10^4 c_2$	$10^4 c_3$	$10^4 (S_M - 1)$
0	cubic	$-1(1)$	$0.12(1)$	$0(2)$
	diamond	$-1(1)$	$0.10(1)$	$0(1)$
	g-expansion	$-3.29, -3.63$	$0.108, 0.102$	$-2.95, -3.50$
	ϵ-expansion	$-2.48, -4.26$	$0.065, 0.114$	$-2.55, -4.38$
1	cubic	$-3.0(2)$	$0.10(1)$	$-2.5(1.0)$
	diamond	$-3.0(2)$	$0.10(2)$	$-2.3(4)$
	g-expansion	$-3.92, -4.27$	$0.126, 0.120$	$-3.50, -4.12$
	ϵ-expansion	$-3.06, -4.99$	$0.080, 0.134$	$-3.14, -5.13$
	impr-ϵ-expansion	$-2.80, -3.64$	$0.060, 0.089$	$-2.86, -3.73$
2	cubic	$-3.9(2)$	$0.11(1)$	$-3.5(1.0)$
	diamond	$-4.1(4)$	$0.10(2)$	$-3.5(3)$
	g-expansion	$-4.22, -4.54$	$0.133, 0.128$	$-3.85, -4.40$
	ϵ-expansion	$-3.39, -5.29$	$0.089, 0.142$	$-3.48, -5.44$
3	cubic	$-4.1(1)$	$0.11(2)$	$-4.1(4)$
	diamond	$-4.5(3)$	$0.11(3)$	$-4.0(4)$
	g-expansion	$-4.29, -4.58$	$0.134, 0.128$	$-3.96, -4.45$
	ϵ-expansion	$-3.56, -4.55$	$0.094, 0.144$	$-3.66, -5.50$
4	cubic	$-4.1(2)$	$0.12(1)$	$-4(1)$
	diamond	$-4.7(2)$	$0.10(2)$	$-4.2(4)$
	g-expansion	$-4.21, -4.46$	$0.130, 0.125$	$-3.92, -4.34$
	ϵ-expansion	$-3.64, -5.28$	$0.096, 0.143$	$-3.74, -5.43$
	$1/N$-expansion	-11.12	0.336	-11.48
8	cubic	$-3.5(1)$	$0.09(2)$	$-3.8(5)$
	diamond	$-4.0(1)$	$0.05(5)$	$-3.8(4)$
	g-expansion	$-3.60, -3.72$	$0.108, 0.103$	$-3.44, -3.68$
	ϵ-expansion	$-3.48, -4.55$	$0.093, 0.124$	$-3.58, -4.68$
	$1/N$-expansion	-5.56	0.118	-5.74
16	cubic	$-2.4(1)$	$0.06(1)$	$-2.8(2)$
	diamond	$-2.65(5)$	$0.05(3)$	$-2.7(3)$
	g-expansion	$-2.46, -2.49$	$0.072, 0.069$	$-2.43, -2.52$
	ϵ-expansion	$-2.73, -3.19$	$0.074, 0.088$	$-2.81, -3.28$
	$1/N$-expansion	-2.78	0.084	-2.87
32	cubic	$-1.45(5)$	$0.04(1)$	$-1.8(3)$
	diamond	$-1.50(5)$	$0.04(1)$	$-1.7(3)$
	g-expansion	$-1.427, -1.429$	$0.041, 0.040$	$-1.45, -1.48$
	ϵ-expansion	$-1.73, -1.84$	$0.047, 0.052$	$-1.78, -1.90$
	$1/N$-expansion	-1.39	0.042	-1.43
∞		0	0	0

TABLE VI. Three-dimensional O(N) σ model with nearest-neighbor interactions: lowest moments of $G(x)$ at $N = \infty$ on the cubic and diamond lattice.

moments	cubic	f.c.c.	diamond
χ	$\frac{1}{\beta z}$	$\frac{1}{2\beta z}$	$\frac{3}{2\beta z}$
$\overline{m}_2$	$\frac{6}{z}$	$\frac{6}{z}$	$\frac{6}{z}$
M_G^2	z	z	z
$\overline{q}_{3,0}$	0	0	$\frac{1}{6\sqrt{3}}\left(1+\frac{z}{12}\right)^{-1}$
$\overline{m}_4$	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$	$\frac{120}{z^2}\left(1+\frac{z}{20}\right)$
$\overline{q}_{4,0}$	$\frac{12}{5z}$	$-\frac{3}{5z}$	$-\frac{8}{5z}$
$\overline{q}_{3,1}$	0	0	$\sqrt{3}\left(1+\frac{z}{18}+\frac{z^2}{216}\right)\left(1+\frac{z}{12}\right)^{-2}$
$\overline{m}_6$	$\frac{5040}{z^3}\left(1+\frac{z}{10}+\frac{z^2}{840}\right)$	$\frac{5040}{z^3}\left(1+\frac{z}{10}+\frac{z^2}{840}\right)$	$\frac{5040}{z^3}\left(1+\frac{11z}{60}+\frac{8z^2}{945}+\frac{z^3}{10080}\right)$
$\overline{q}_{4,1}$	$\frac{528}{5z^2}\left(1+\frac{z}{44}\right)$	$-\frac{132}{5z^2}\left(1+\frac{z}{44}\right)$	$-\frac{352}{5z^2}\left(1+\frac{z}{33}+\frac{z^2}{528}\right)\left(1+\frac{z}{12}\right)^{-1}$
$\overline{q}_{6,0}$	$\frac{12}{77z}$	$-\frac{39}{154z}$	$-\frac{416}{231z}\left(1-\frac{z}{78}\right)\left(1+\frac{z}{12}\right)^{-1}$
$\overline{m}_8$	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{11z^2}{2160}+\frac{z^3}{60480}\right)$	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{37z^2}{7560}+\frac{z^3}{60480}\right)$	$\frac{362880}{z^4}\left(1+\frac{3z}{20}+\frac{389z^2}{136080}+O(z^3)\right)$
$\overline{q}_{4,2}$	$\frac{41184}{5z^3}\left(1+\frac{19z}{286}+\frac{z^2}{3432}\right)$	$-\frac{10296}{5z^3}\left(1+\frac{9z}{143}+\frac{z^2}{3432}\right)$	$-\frac{27456}{5z^3}\left(1+\frac{z}{858}+\frac{139z^2}{30888}+O(z^3)\right)$
$\overline{q}_{6,1}$	$\frac{240}{11z^2}\left(1+\frac{z}{140}\right)$	$-\frac{1110}{77z^2}\left(1+\frac{13z}{740}\right)$	$-\frac{23680}{231z^2}\left(1-\frac{129z}{1480}+\frac{5z^2}{444}+O(z^3)\right)$

TABLE VII. Two-dimensional O(N) σ model with nearest-neighbor interactions: lowest moments of $G(x)$ at $N = \infty$ on the square, triangular, and honeycomb lattice.

moments	square	triangular	honeycomb
χ	$\frac{1}{\beta z}$	$\frac{2}{3\beta z}$	$\frac{4}{3\beta z}$
$\overline{m}_2$	$\frac{4}{z}$	$\frac{4}{z}$	$\frac{4}{z}$
M_G^2	z	z	z
$\overline{t}_{3,0}$	0	0	$\frac{1}{2}\left(1+\frac{z}{8}\right)^{-1}$
$\overline{m}_4$	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$	$\frac{64}{z^2}\left(1+\frac{z}{16}\right)$
$\overline{q}_{4,0}$	$\frac{1}{z}$	0	0
$\overline{m}_6$	$\frac{2304}{z^3}\left(1+\frac{z}{8}+\frac{z^2}{576}\right)$	$\frac{2304}{z^3}\left(1+\frac{z}{8}+\frac{z^2}{576}\right)$	$\frac{2304}{z^3}(1+\frac{z}{4}+\frac{z^2}{64}+\frac{z^3}{4608})(1+\frac{z}{8})^{-1}$
$\overline{q}_{4,1}$	$\frac{40}{z^2}\left(1+\frac{z}{40}\right)$	0	0
$\overline{t}_{6,0}$	0	$-\frac{4}{z}$	$\frac{36}{z}\left(1-\frac{z}{72}\right)\left(1+\frac{z}{8}\right)^{-1}$

TABLE VIII. For various values of N, we report estimates of σ obtained by our strong-coupling analysis, from the $1/N$-expansion, from the resummation of the g-expansion (see Section III D) (in this case we give two numbers corresponding to the two choices: resumming $R(x)$ or $R(x)/x^2$), and from the $O(\epsilon^2)$ term of the ϵ-expansion. In order to derive σ from $\sigma\nu$, which is what is computed in the strong-coupling analysis, we used the following values of ν: $\nu \simeq 0.59$ for $N = 0$; $\nu \simeq 0.63$ for $N = 1$; $\nu \simeq 0.67$ for $N = 2$; $\nu \simeq 0.71$ for $N = 3$; $\nu \simeq 0.74$ for $N = 4$; $\nu \simeq 0.83$ for $N = 8$; $\nu \simeq 0.91$ for $N = 16$; $\nu \simeq 0.96$ for $N = 32$. The errors we report for the strong-coupling estimates take into account all the analyses we performed.

N	s.c.-expansion	$1/N$-expansion	g-expansion	ϵ-expansion
0	0.00(1)		0.0119, 0.0141	0.0109
1	0.01(1)		0.0143, 0.0166	0.0130
2	0.02(1)		0.0156, 0.0177	0.0140
3	0.03(2)	0.0515	0.0160, 0.0179	0.0145
4	0.03(2)	0.0386	0.0158, 0.0174	0.0147
8	0.02(1)	0.0193	0.0139, 0.0148	0.0137
16	0.009(3)	0.0096	0.0098, 0.0109	0.0109
32	0.004(2)	0.0048	0.0058, 0.0059	0.0074

TABLE IX. Estimates of σ_6, obtained by applying the CPRM to the series $q_{6,0}$ and m_2 on the cubic lattice. The errors reported in the Table take into account all the analyses we performed.

N	s.c.-expansion	$1/N$-expansion	ϵ-expansion
0	0.01(1)		0.0134
1	0.03(2)		0.0159
2	0.04(2)		0.0171
3	0.04(2)		0.0177
4	0.036(10)	0.0491	0.0178
8	0.024(4)	0.0245	0.0167
16	0.013(2)	0.0123	0.0134
32	0.0065(8)	0.0061	0.0091

TABLE X. Estimates of d_1 from the analysis of the strong-coupling series on the cubic and diamond lattice. The last column is our final estimate. An asterisk indicates that most of the approximants we considered are defective, or, in the cases where numbers are not shown, that all the approximants are defective, so that no estimate can be extracted. We also report results obtained by resumming the available terms of the g-expansion, from the $1/N$ calculation.

N		10^4PA	10^4DPA	10^4 IA	$10^4 d_1$
0	cubic	1.3(7)	*	*0.4(9)	1(1)
	diamond	−0.9(5)	*	*−1.2(2)	−1.0(5)
	g-expansion				−1.31, −1.60
1	cubic	−1.6(8)	*	*−1.7(1)	−1.7(5)
	diamond	−3(1)	−3.1(7)	*−3.1(3)	−3(1)
	g-expansion				−1.59, −1.89
2	cubic	−2.2(3)	*	−2.3(1)	−2.3(2)
	diamond	−3(1)	−2.8(3)	*−3.7(9)	−3(1)
	g-expansion				−1.72, −2.01
3	cubic	−2.4(3)	*	−2.5(1)	−2.5(2)
	diamond	−2.2(9)	−2.6(3)	−2.6(7)	−2.6(3)
	g-expansion				−1.77, −2.03
4	cubic	−2.4(3)	*	−2.5(2)	−2.5(2)
	diamond	−5(3)	−2.3(3)	*−4(3)	−2.5(5)
	g-expansion				−1.76, −1.99
	$1/N$-expansion				−5.12
8	cubic	*-2.0(4)	*	−2.1(2)	−2.1(2)
	diamond	−5(3)	−2(2)	*−4(1)	−3(2)
	g-expansion				−1.55, −1.68
	$1/N$-expansion				−2.56
16	cubic	−1.3(3)	*	−1.4(2)	−1.4(2)
	diamond	−3(3)	−1.2(8)	*	−1.2(8)
	g-expansion				−1.10, −1.15
	$1/N$-expansion				−1.28
32	cubic	−0.7(3)	*	−0.7(3)	−0.7(2)
	diamond	−1(1)	−0.4(2)	*	−0.5(3)
	g-expansion				−0.66, −0.67
	$1/N$-expansion				−0.64
∞					0

TABLE XI. Coefficients of the strong-coupling expansion of $G^{-1}(x)$ on the cubic lattice. The representative of each equivalence class is chosen by $x_1 \geq x_2 \geq x_3 \geq 0$. l indicates the order.

x_1	x_2	x_3	l	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$	$N=16$
0	0	0	0	1	1	1	1	1	1
1	0	0	1	-1	-1	-1	-1	-1	-1
0	0	0	2	6	6	6	6	6	6
1	0	0	3	-1	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{1}{9}$
0	0	0	4	30	26	24	$\frac{114}{5}$	22	$\frac{58}{3}$
1	0	0	5	-13	$-\frac{122}{15}$	$-\frac{35}{6}$	$-\frac{158}{35}$	$-\frac{11}{3}$	$-\frac{49}{45}$
0	0	0	6	366	$\frac{4204}{15}$	$\frac{479}{2}$	$\frac{37788}{175}$	$\frac{602}{3}$	$\frac{20674}{135}$
1	1	0	6	2	0	$-\frac{1}{2}$	$-\frac{16}{25}$	$-\frac{2}{3}$	$-\frac{10}{27}$
1	0	0	7	-197	$-\frac{33604}{315}$	$-\frac{1123}{16}$	$-\frac{44812}{875}$	$-\frac{1793}{45}$	$-\frac{132961}{13365}$
0	0	0	8	5022	$\frac{348266}{105}$	$\frac{10265}{4}$	$\frac{1882494}{875}$	$\frac{9454}{5}$	$\frac{5012914}{4455}$
1	1	0	8	24	-16	$-\frac{65}{3}$	$-\frac{18576}{875}$	$-\frac{176}{9}$	$-\frac{1912}{243}$
2	0	0	8	4	0	-1	$-\frac{32}{25}$	$-\frac{4}{3}$	$-\frac{20}{27}$
1	0	0	9	-2889	$-\frac{3805202}{2835}$	$-\frac{191503}{240}$	$-\frac{5197194}{9625}$	$-\frac{53443}{135}$	$-\frac{2940271}{40095}$
1	1	1	9	6	-8	$-\frac{57}{4}$	$-\frac{408}{25}$	$-\frac{50}{3}$	$-\frac{778}{81}$
2	1	0	9	-1	0	$-\frac{1}{4}$	$-\frac{56}{125}$	$-\frac{5}{9}$	$-\frac{115}{243}$
0	0	0	10	76062	$\frac{211434604}{4725}$	$\frac{3833513}{120}$	$\frac{8484905796}{336875}$	21078	$\frac{1969903274}{200475}$
1	1	0	10	258	$-\frac{952}{3}$	$-\frac{24851}{72}$	$-\frac{1878984}{6125}$	$-\frac{35786}{135}$	$-\frac{10005158}{120285}$
2	0	0	10	116	-24	$-\frac{583}{12}$	$-\frac{45272}{875}$	$-\frac{1340}{27}$	$-\frac{49060}{2187}$
1	0	0	11	-45357	$-\frac{2874597004}{155925}$	$-\frac{42875903}{4320}$	$-\frac{949030894596}{153278125}$	$-\frac{12000923}{2835}$	$-\frac{34414485049}{70366725}$
1	1	1	11	72	-240	$-\frac{2635}{8}$	$-\frac{7370064}{21875}$	$-\frac{8636}{27}$	$-\frac{915928}{6561}$
2	1	0	11	-15	-24	$-\frac{133}{3}$	$-\frac{1142304}{21875}$	$-\frac{4369}{81}$	$-\frac{605717}{19683}$
3	0	0	11	0	0	$-\frac{1}{2}$	$-\frac{96}{125}$	$-\frac{8}{9}$	$-\frac{160}{243}$
0	0	0	12	1230462	$\frac{101355262012}{155925}$	$\frac{77441167}{180}$	$\frac{49050932335452}{153278125}$	$\frac{26831414}{105}$	$\frac{2184687177202}{23455575}$
1	1	0	12	2460	$-\frac{254288}{45}$	$-\frac{470591}{90}$	$-\frac{49887963408}{11790625}$	$-\frac{1379716}{405}$	$-\frac{3960090404}{5412825}$
2	0	0	12	1944	-896	$-\frac{43417}{36}$	$-\frac{35397888}{30625}$	$-\frac{423976}{405}$	$-\frac{427970056}{1082565}$
2	1	1	12	0	-24	$-\frac{393}{8}$	$-\frac{39096}{625}$	$-\frac{616}{9}$	$-\frac{35456}{729}$
2	2	0	12	0	0	$-\frac{7}{8}$	$-\frac{1104}{625}$	$-\frac{64}{27}$	$-\frac{5600}{2187}$
3	1	0	12	0	0	$-\frac{1}{8}$	$-\frac{192}{625}$	$-\frac{4}{9}$	$-\frac{400}{729}$
1	0	0	13	-745189	$-\frac{1629512844964}{6081075}$	$-\frac{2652479497}{20160}$	$-\frac{11642408503972}{153278125}$	$-\frac{45795581}{945}$	$-\frac{470143113389}{164189025}$
1	1	1	13	-678	$-\frac{28728}{5}$	$-\frac{614743}{96}$	$-\frac{903833736}{153125}$	$-\frac{704326}{135}$	$-\frac{600662842}{360855}$
2	1	0	13	-476	-1120	$-\frac{437317}{288}$	$-\frac{241988968}{153125}$	$-\frac{616348}{405}$	$-\frac{765424036}{1082565}$
3	0	0	13	0	-24	$-\frac{1337}{24}$	$-\frac{1500984}{21875}$	$-\frac{5864}{81}$	$-\frac{868672}{19683}$
0	0	0	14	20787102	$\frac{140597688722408}{14189175}$	$\frac{245838046393}{40320}$	$\frac{298587843693288}{69671875}$	$\frac{46330187578}{14175}$	$\frac{15116615635191578}{16254713475}$
1	1	0	14	17378	$-\frac{18013984}{189}$	$-\frac{1330017637}{17280}$	$-\frac{43519901193056}{766390625}$	$-\frac{1801726502}{42525}$	$-\frac{35538833424418}{6966305775}$
2	0	0	14	29088	-20744	$-\frac{8258723}{360}$	$-\frac{1192015283352}{58953125}$	$-\frac{21049024}{1215}$	$-\frac{253843671008}{48715425}$
2	1	1	14	0	-1456	$-\frac{54377}{24}$	$-\frac{278755152}{109375}$	$-\frac{628336}{243}$	$-\frac{253847360}{177147}$
2	2	0	14	32	-144	$-\frac{2425}{8}$	$-\frac{42568048}{109375}$	$-\frac{103640}{243}$	$-\frac{52536352}{177147}$
3	1	0	14	-2	-32	$-\frac{206}{3}$	$-\frac{1427216}{15625}$	$-\frac{8294}{81}$	$-\frac{4477778}{59049}$
4	0	0	14	0	0	$-\frac{1}{4}$	$-\frac{288}{625}$	$-\frac{16}{27}$	$-\frac{1280}{2187}$
1	0	0	15	-12672757	$-\frac{2597638257068408}{638512875}$	$-\frac{3549389785799}{1935360}$	$-\frac{64161784918165784}{65143203125}$	$-\frac{25033873577}{42525}$	$-\frac{305112195055173211}{24138249510375}$
1	1	1	15	-48624	$-\frac{7317712}{63}$	$-\frac{317601607}{2880}$	$-\frac{555516678768}{6015625}$	$-\frac{92268604}{1215}$	$-\frac{2363546376656}{146146275}$
2	1	0	15	-16428	$-\frac{154436}{5}$	$-\frac{605816971}{17280}$	$-\frac{1403163936156}{42109375}$	$-\frac{110231638}{3645}$	$-\frac{4917927730652}{438438825}$
2	2	1	15	30	-232	$-\frac{8199}{16}$	$-\frac{2141976}{3125}$	$-\frac{20954}{27}$	$-\frac{4066162}{6561}$
3	0	0	15	-400	-1752	$-\frac{192203}{72}$	$-\frac{11182524776}{3828125}$	$-\frac{10592632}{3645}$	$-\frac{133926221584}{87687765}$
3	1	1	15	-10	-48	$-\frac{1681}{16}$	$-\frac{90832}{625}$	$-\frac{506}{3}$	$-\frac{953338}{6561}$
3	2	0	15	-1	0	$-\frac{13}{8}$	$-\frac{11936}{3125}$	$-\frac{151}{27}$	$-\frac{50545}{6561}$
4	1	0	15	0	0	$-\frac{1}{16}$	$-\frac{648}{3125}$	$-\frac{28}{81}$	$-\frac{11840}{19683}$